AN INTRODUCTION TO FLUXIONS,

Deſigned for the Uſe,

AND ADAPTED TO THE CAPACITIES OF BEGINNERS.

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> An Introduction ought to have two Properties; the one, that of a perſpicuous and clear Method; and the other, that of an univerſal Latitude and Comprehenſion; where the Students may have a little pre-notion of every thing, like a Model towards a great Building.
>
> LORD BACON.

LONDON:
Printed for J. NOURSE, Bookſeller to His MAJESTY.

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THE

PREFACE.

I AM well aſſured, that neither a pompous Title, nor a long Preface, can confer any real value on a Book, if it has nothing in it to recommend it; yet ſince it is now become cuſtomary for an Author to give an account of his undertaking, I ſhall readily comply therewith, leſt I ſhould be thought to be as careleſs and indifferent to my Readers, as the man was to his *Prince*, who went to *Court* without his Cravat.

The deſign of writing this ſhort Introduction was for beginners, in order to render the entrance into this moſt ſublime Geometry plain, eaſy, and delightful as poſſible; which is generally very intricate and narrow, tho' the ſubject is admirable, and every part of it uſeful and entertaining. As the entrance into this Science will always be recommended to beginners with better ſucceſs, when its Principles are explained by a variety of Examples in an elementary way, than by any other; ſo I hope I have hit upon a method, better adapted to beginners, by reaſon of its plainneſs and perſpicuity, than has hitherto appeared; wherein I have endeavoured not to puzzle and confound, but to make things plain and practicable, and have omitted nothing that might be materially uſeful, with regard to the Limits of this Introduction.

The Book is divided into four Sections, as follows. In the *Firſt* are given the generation of Powers, and notation of Indices or Exponents, of which I have treated largely, which will be found very uſeful in *infinite Series* and *Fluxions*, the want of knowing how to expreſs quantities by Indices, I have often found to be a great hindrance to beginners.

The *Second* Section treats of *Infinite Series*, that is, ſhewing the Method how to put an Expreſſion into a Series, whether it be whole or a fraction, affirmative or negative. 1. By Diviſion. 2. By Extraction of Roots. 3. By the Binomial Theorem, wherein the Algorithm of Indices are greatly illuſtrated. The operations at large are annexed, where there ſeems to want any explanation; and in order to prove the truth of the inveſtigations, I have ſhewn how to reſtore the Original Powers, whether by Diviſion, or Extraction, whereby the Learner will be enabled of himſelf to raiſe any multinomial to any given power, or to find the root of the ſame. I have alſo ſhewn the uſe of Sir *Iſaac Newton*'s famous Binomial Theorem, by a great variety of Examples; to all which I have ſubjoined a method for obtaining a Formula, for raiſing an infinitinomial to any power, and ſix Theorems in Reverſion of Series.

In the *Third* Section are laid down the Firſt Principles of *Fluxions* according to the Inventor Sir *Iſaac Newton*, treating both of the *Direct* and *Inverſe Method*; the Rules are illuſtrated with a great variety of Examples, as well for putting Expreſſions into Fluxions, with ſeveral univerſal Theorems, as thoſe for finding of *Fluents*. To which are annexed ſeveral general Propoſitions for finding and comparing Fluents, and one communicated to me by that ready Analyſt Mr. *Powle* of *Hereford*.

The *Fourth* Section ſhews a brief application of Fluxions *in Maximis & Minimis*; drawing Tangents to Curves; finding the length of Curve Lines; the Areas of Curves; the Surfaces of Solids, generated by the Rotation of Planes about an Axis;

the Contents of Solid Bodies, to which are added above forty Fluxionary Problems with their Solutions at large.

In the year 1745, when I was in London, in company with the celebrated Mathematicians *W. Jones*, Eſq; and Mr. *De Moivre*, they expreſſed their great approbation of Mr. *Emerſon*'s Book of Fluxions, then lately publiſhed, with regard to the method and neatneſs of the manner of his treating the ſubject, but were of opinion he began his book too high for beginners, and that his 4, 5, &c. Propoſitions were very elegant and uſeful in comparing of Fluents, but too difficult and abſtracted; I have therefore undertaken to explain them, by ſeveral familiar Examples, which the Reader will find in this work. The reaſon I ſuppoſe why Mr. *Emerſon* began his ſubject high, was deſignedly; for no doubt, he conſidered that all authors that wrote upon Fluxions, ſpent a great deal of words in explaining the firſt Principles, and ſometimes made them rather more obſcure by a multitude of words, and beſides, laid the ſubject ſo low, and upon ſuch incorrect principles, that the Science could not be defended againſt the Cavils of ſuch as were profeſſed Enemies to Science. He was determined to ſtop the mouth of theſe Metaphyſical Critics, that thought they could overturn the whole foundation thereof by their ſubtle diſputings. In order to this, he endeavoured to lay the foundation with all the accuracy and ſtrictneſs poſſible, and upon ſuch principles as could not be eluded by all the art and ſophiſtry of theſe ſort of men. This being done, he haſtened with all ſpeed to raiſe the ſuperſtructure, paſſing more ſlightly over ſuch things as had been more fully treated on by ſome, and ſpending more time upon ſuch things as had been paſſed ſlightly over by others; and dwelling largely upon ſuch things as were wanting. But he never was ſo vain, as to think that any perſon who is a reader of Fluxions, would be content with his book alone; (as if no Flowers of Science could grow in another Garden,) ſo that if they found themſelves at a loſs in the firſt principles in reading his book, as being perhaps too abſtracted, they might receive abundant ſatisfaction in every common author, but his book has met with the approbation of the Learned, ſo as to paſs two Editions.

In laying down the firſt principles of Fluxions, I have adhered cloſely to the noble Inventor Sir *Iſaac Newton*, and have in the general Definition conſulted the Latin Comment on the *Principia*; but the ſame things are briefly laid down without uſing any infinitely ſmall quantities, or vaneſcent Parallelograms; which perhaps will be more acceptable to many of my Readers. In the firſt general Definition, the Reader will perceive that there are quantities called *naſcent* and *evaneſcent*, which are here words of great propriety. *Naſcent* quantities, or *naſcent Increments*, are thoſe when they *firſt begin* to be, and which is daily applied to objects during the time they *firſt begin* to appear: as for inſtance, the flowing of a point generates a line, the very moment that the generating point *begins* to move, at that very moment an increment *begins* to be generated, this ariſing quantity is called a naſcent increment, or naſcent quantity. An *Evaneſcent* quantity, or rather *vaniſhing* quantity (for the word *evaneſcens* is a participle, and not a noun adjective) is ſuch, that there is aſcribed to it a perpetual diminution; that is, from that time the quantity is *going* to vaniſh.

Sir *Iſaac Newton* has accompliſhed by inventing the method of Fluxions, what *Cavallerius* wiſhed for, and offering it in a way that admits of ſtrict demonſtration, which requires the ſuppoſition of no quantities but ſuch as are finite and eaſily conceived;

ceived; by his Doctrine of Prime and Ultimate Ratios, he has found out the proper medium whereby to avoid the impoſſible notion of *Indiviſibles* on the one hand, and the length of *Exhauſtions* on the other. Perhaps it may be objected, ſays *Sir Iſaac*, that there is no ultimate proportion of evaneſcent quantities, becauſe the proportion, before the quantities have vaniſhed, is not the ultimate, and when they are vaniſhed, are none. But by the ſame argument it may be alledged, that a body arriving at a certain place, and there ſtopping, has no ultimate velocity; becauſe the velocity, before the body comes to the place, is not its ultimate velocity, when it has arrived is none. But the anſwer is eaſy, ſays *Sir Iſaac*; for by the Ultimate Velocity is meant *with which* the body is moved, neither *before* it arrives at its laſt place and the motion ceaſes, nor *after*, but at the *very inſtant* it arrives; that is, that velocity *with which* the body arrives at its laſt place, and *with which* the motion ceaſes. And in like manner, by the ultimate ratio of evaneſcent quantities is to be underſtood the ratio of the quantities not *before* they vaniſh nor *afterwards*, but *with which* they vaniſh. In like manner the prime or firſt ratio of naſcent quantities is that with which they *begin* to be. And the firſt or laſt ſum is that with which they *begin* and *ceaſe* to be, (or to be augmented or diminiſhed.) There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the Ultimate Velocity. And there is the like limit in all quantities and proportions that *begin* and *ceaſe* to be. And ſince ſuch limits are certain and definite, to determine the ſame is a problem ſtrictly geometrical. In all this plain narrative of matter of fact, there appears no inconſiſtency in Sir *Iſaac Newton*'s account of his methods, or leaſt ſhadow of his having been ever confounded in his ideas about them; as has been miſrepreſented by thoſe who find fault with him, for want of their rightly underſtanding the nature of theſe great inventions. This has been the grounds on which they will not allow Sir *Iſaac* to have any great talent of explaining himſelf; and indeed this has been the opinion of ſeveral, but it is certainly a very great miſtake; for he is the cleareſt writer for the number of words he uſes of any man that ever writ; he has no waſte words, nor does he want any, if one does but attend carefully to the ſtrict ſenſe of the few he uſes; but the true reaſon of his not being underſtood is really the difficulty of the ſubject, or perhaps his readers being ſometimes not duly qualified.

This Prince of Philoſophers was the firſt, who introduced Indefinite Indices or Exponents of Powers into analytical operations; he firſt taught to reduce Fractions and Radicals and the adfected Roots of Equations, or any Power of a Binomial into a converging Series to ſquare the Curve. He laid down the idea of deducing the area from the ordinate by conſidering the area as a naſcent quantity, growing or increaſing by continual flux, in proportion to the length of the ordinate; and ſuppoſing the abſciſſa to increaſe uniformly in proportion to the time; and from the moments of the time he gave the name of Moments to the momentaneous increments or infinitely ſmall parts of the abſciſſa and area, generated in moments of time; the moment of a line he called a Point, in the ſenſe of *Cavallerius*, though it be not a geometrical point, but a line infinitely ſhort; and the moment of an area or ſuperficies, he called a line in the ſenſe of the ſame *Cavallerius*, though it be not a geometrical line, but a ſuperficies infinitely narrow; and when he conſidered the ordinate as the moment of the area, he underſtood by it the rectangles under the geometrical ordinate, and a moment of the abſciſſa;

ſciſſa, though that moment be not always expreſſed. Sir *Iſaac* in his Compendium repreſents the uniform fluxion of time, or of any exponent of time, by an unit; the moment of time, or of its exponent by the letter o; the Fluxions of other quantities by any other ſymbols; the moments of thoſe quantities, by the rectangle under thoſe ſymbols and the letter o; the area of a curve he puts for a Fluent, and the ordinate for its Fluxion: when he is demonſtrating any propoſition, he uſes the letter o, for a finite moment of time, or of its exponent, or of any quantity flowing uniformly, and performs the calculation by the geometry of the ancients in finite figures, or ſchemes without any approximation; and as ſoon as the calculation is at an end, and the equation is reduced, he ſuppoſes the moment o to be infinitely ſmall, and forbears to write it down, and uſes all manner of approximation, which he conceives will produce no error in the concluſion.

He ſays, in his *Analyſis*, *Momentum eſt ſuperficies cum de ſolidis, et linea cum de ſuperficiebus, et punctum cum de lineis agitur*, i. e. that when ſolids are conſidered as Fluents, their moments are ſuperficies, and the moment of thoſe moments or ſecond moments are lines, and the moments of thoſe moments or third moments are points in the ſenſe of *Cavallerius*; and in his *Principia*, where he frequently conſiders lines as Fluents, deſcribed by points, whoſe velocities increaſe or decreaſe; the velocities are the firſt Fluxions, and their increaſe the ſecond. The method of Fluxions has all the advantages of any others, and is more elegant, becauſe in this calculation there is but one infinitely ſmall quantity repreſented by a ſymbol, which is o; we have no ideas of infinitely ſmall quantities, and therefore Sir *Iſaac* introduced Fluxions into his method, that it might proceed by finite quantities as much as poſſible; it is more natural and geometrical, becauſe founded upon prime ratios of naſcent quantities, which have a being in Geometry, whilſt *Indiviſibles* on the contrary have no being either in Geometry or nature; there are, it is true, prime ratios of naſcent quantities, but no prime naſcent quantities. Nature generates quantities by continual flux, or increaſe, and the ancient Geometricians admitted ſuch a generation of areas and ſolids, when they drew one line into another by local motion to generate a ſurface or a ſolid, but the ſumming up of *Indiviſibles* to compoſe an area or ſolid was never yet admitted into Geometry. The method of Fluxions is alſo of greater uſe and certainty, being adapted either to the ready finding out of a propoſition by ſuch approximations, as will create no error in the concluſion, or, to the demonſtrating it exactly. By the help of this new *Analyſis*, Sir *Iſaac* found out moſt of the propoſitions in his *Principia*, but becauſe the ancients, for making things certain, admitted nothing into Geometry before it was demonſtrated ſynthetically, He therefore demonſtrated the propoſitions ſynthetically in that Book, that the ſyſtem of the Heavens might be founded upon good Geometry; and this makes it difficult for the unſkilful to ſee the Analyſis by which thoſe propoſitions were found out.

In 1665, Dr. *Wallis* publiſhed his *Arithmetica Infinitorum*, in which he ſquared a ſeries of curves, whoſe ordinates were $\overline{1-x^2}|^{1}$. $\overline{1-x^2}|^{2}$. $\overline{1-x^2}|^{3}$. $\overline{1-x^2}|^{4}$, &c. and ſhewed, that if the ſeries of their areas could be interpoled in the middle places, the interpolation would give the quadrature of the circle. Sir *Iſaac* (then Mr.) *Newton*, in the winter between the years 1664 and 1665 (when he was only twenty-two years of age) upon the reading of that Book, and conſidering how to interpole the ſeries of the areas, found that the area of the circular ſector ſtanding upon the arc, whoſe ſine is x, the radius being unity, might be expreſſed by this ſeries, $x - \frac{1}{6}x^3 - \frac{1}{46}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9$, &c. and from hence he readily derived the ſeries $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9$, &c. for the length of the arc whoſe ſine is x, from no other conſideration than that the arc has the ſame proportion to its ſector, as the whole quadrant has to an arc of 90 degrees. At the ſame time, and in the ſame manner, he found the ſeries $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$, &c. to be the hyperbolic area in the rectangular hyperbola intercepted between the curve, its aſſymptote, and two ordinates, whoſe diameter is x parallel to the other aſſymptote.

In the ſummer, 1665, being obliged to retire from *Cambridge* on account of the Plague, he went to *Boothby* in *Lincolnſhire*, where he computed the area of the Hyperbola by this ſeries to two and fifty figures. In the ſame ſeaſon of that year, by means of an happy thought of ſubſtituting indefinite indices of powers, he put the 59th Propoſition of that Book, which Dr. *Wallis* arrived at by various ſteps, in another language, and moſt general form, in this manner; if the Abſciſſa of any curvilinear figure be called x, let m and n be numbers, and the ordinate erected at right angles be $x^{\frac{m}{n}}$, the area of the figure ſhall be $\frac{n}{m+n}x^{\frac{m+n}{n}}$, and if the ordinate be compoſed of two ſigns + and —, the area will be compoſed of two or more ſuch areas taken with their ſigns + or —. In the beginning of the year 1665, he found a method of Tangents like that of *Hudden*, *Gregory*, or *Sluſius*, and a method of finding the Crookedneſs of Curves at any given point. And continuing to purſue the method of interpolation, he found the quadrature of all Curves, whoſe ordinates are the powers of Binomials affected with indices, whole, fracted, or ſurd, affirmative or negative, together with a rule for reducing any power of a Binomial into an approximating or converging ſeries.' For by interpoling the ſeries of the powers of a Binomial $a + x . a^2 + 2ax + x^2 . x^3 + 3a^2x + 3ax^2 +$ x^3, &c. he diſcovered that $\overline{a+x}|^n = a^n + na^{n-1}x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2}x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}x^3 +$ &c. where the index (n) of the power may alſo be any number, whole, or fracted, or ſurd, affirmative or negative, a and x any quantities. In the ſpring of that year, he found how to perform the ſame by continual diviſion and extraction of roots. Soon after this he extended the method to extraction of the roots of adfected equations in ſpecies. And he was the firſt who introduced into *Analyſis*, Fractions, negative or indefinite quantities for the indices or exponents of powers; and by that means reduced the operations of multiplication, diviſion and extraction of roots to one common way of conſidering them, and thereby very much enlarged the bounds of *Analyſis*, and laid the foundation of making it univerſal. About three years after, that is in 1668, the Lord Viſcount *Brouncker* publiſhed the quadrature of the hyperbola by this ſeries $\frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10}$, &c. which is no other than Mr. *Newton's* ſeries which he diſcovered before, viz. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$, &c. and ſoon after, this quadrature of the hyperbola was demonſtrated by *Nicholas Mercator*, by means of the diviſion firſt made uſe of by Dr. *Wallis* in his *Opus Arithmeticum* publiſhed in the year 1657, where the fraction $\frac{a}{1+r}$ is reduced by perpetual diviſion into the ſeries $a + ar + ar^2 + ar^3 + ar^4$, &c. thus we may obſerve how far *Mercator* had any pretence to the diſcovery of the Quadrature of the Hyperbola, ſince Dr. *Wallis* had found the Diviſion long before, and alſo the quadrature of every part of the quotient, which *Mercator* ſhould have acknowledged, when he put theſe two inventions together. *Mercator*, who was born in *Holſtein*, but ſpent moſt of his time in *England*, publiſhed his *Logarithmetechnia* in 1688, in which he gave the quadrature of the Hyperbola by an infinite Series. This was the firſt appearance, in the learned world, of a Series of this ſort, drawn from the particular nature of the Curve, and that in a manner very new and abſtracted. Dr. *Barrow*, then at *Cambridge*, where Mr. *Newton* (who was about 26 years of age) reſided, recollected, that he had met with the ſame thing in the writings of that young Gentleman, and there not confined to the Hyperbola only, but extended by general forms to all ſorts of Curves, even ſuch as are mechanical, to their quadratures, their rectifications, and their centers of gravity, to the ſolids formed by their rotations, and to the ſuperficies of thoſe ſolids; ſo that ſuppoſing their determinations to be poſſible, the Series ſtopt at a certain point, or at leaſt their ſums were given by ſtated rules. But if the abſolute determinations were impoſſible, they could yet be infinitely approximated; which is the happieſt and moſt refined method, ſays *Monſieur de Fontenelle*, of ſupplying the defects of human knowledge, that man's imagination could poſſibly invent. To be maſter of ſo fruitful and general a Theory, was a mine of gold to a Geometrician; but it was a greater glory to have been the Diſcoverer of ſo ſurpriſing and ingenious a Syſtem. So that Mr. *Newton* finding by *Mercator*'s book, that he was in the way to it, and that others might follow in his track, ſhould naturally have been forward to open his treaſures, and ſecure the property which conſiſted in making the diſcovery. But he contented himſelf with his treaſure which he had found without regarding the glory. This manuſcript upon Infinite Series was communicated to none but Mr. *John Collins*, F. R. S. and the Lord *Brouncker*; and even this had not been done but for Dr. *Barrow*, who would not ſuffer him to indulge his modeſty ſo much as he deſired. But what is ſtill more ſurpriſing, this manuſcript, which was taken out of his ſtudy in 1669, contains both the admirable Theory of his Series, and the diſcovery and method of Fluxions, when he was not 24 years of age.

Obvia conſpicimus, nubem pellente Matheſi;
Quæ ſuperas penetrare Domos, atque ardua Cœli,
NEWTONI auſpiciis, jam dat contingere templa.
Surgite mortales, terrenas mittite Curas.

HALLEY.

I hope the reader will not be diſpleaſed with this ſhort account, extracted from the *Philoſophical Tranſactions*, *Commercium Epiſtolicum Collinii*, &c. of the noble Inventor of Fluxions, and which moreover ſhews how young he was, when he made theſe great diſcoveries. It perhaps might be wiſhed, that I had carried on my ſubject farther, but I would have this remarked once for all, that I was writing for beginners, and not for thoſe who have made ſome proficiency in Fluxions, and have rather conſulted the limits of the book, than the copiouſneſs of the ſubject, ſeeing this is handled by the univerſally learned Mr. *Emerſon*, the late Profeſſor *Simpſon*, and others, in a much more elegant method, than one who has neither leiſure nor abilities for ſuch an undertaking; neither is it in the leaſt pretended that every thing is here new, yet I hope the method, I have treated it in, will appear to be clear and intelligible, ſimple and eaſy to be apprehended by thoſe who are not able to underſtand the more learned Authors; neither do I preſume to compare this performance with theirs who are more correct, or detract in the leaſt from their juſt value, to give this any advantage; my buſineſs was only to pave the way a little to thoſe who have wrote profeſſedly on the Doctrine of Fluxions, and I hope this will be found an uſeful Introduction thereto, in order for the reading of thoſe. At the concluſion of the Book I have inſerted ſeveral Problems ſhewing the uſe of Fluxions, ſome of which are from the Diaries, and my *Miſcellanea Curioſa Mathematica*, but the Solutions are entirely new, and I think more eaſily to be underſtood, by any one tolerably verſed in Arithmetic and Common Algebra.

But perhaps what I have done, may be looked upon in this mercenary age, (where men regard *Pluto* more than *Minerva*,) as a meer amuſement or an uſeleſs ſpeculation, eſpecially in this part of the country, where theſe Sciences are not at all regarded, the diſlike ariſing, I preſume, from men's prejudice or ignorance of them thro' their education; but it is a meer trifling excuſe for the *Great* to ſay they don't underſtand theſe Sciences, who really ſhould be Patrons and Promoters of them; they would then grow into ſome eſteem; and we might hope to ſee united in character, what has been ſometimes thought incompatible, the *Gentleman* and the *Geometrician*.

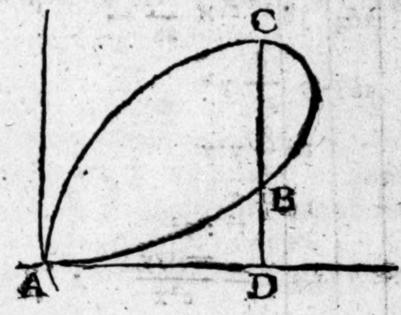
The ſtudy of the Mathematics has in all ages and countries where learning hath prevailed, been regarded as one of the moſt valuable parts of Science, it accuſtoms the mind to reflection, and gives it a habit of cloſe and demonſtrative reaſoning. This ſtudy is highly conducive to the Divine, the Phyſician, and the Orator, for it conduces much to the improvement of the underſtandings and reaſon of men, by enabling them to expreſs the truth of Nature and Religion, and give a juſt repreſentation of things as they are in themſelves; and not give the mind the liberty of transforming them into what ſhapes we pleaſe, and never bring them forth to view but in ſome diſguiſe of our own; and this too often happens, that men, ignorant of the Mathematics, are never more ſuperficial, than when they pretend to go to the bottom of things. This ſtudy is particularly the baſis of thoſe ſciences, which give us a clear and extenſive knowledge of the ſyſtem of the world, and as they lead us to the moſt profound reverence of the ALMIGHTY and wiſe *Creator*, ſo they free us from the mean and narrow thoughts that ſpring from prejudice, ſuperſtition and bigotry. In ſhort, the ſtudy of the Mathematics is not only an agreeable and beneficial amuſement in a retired and country life, but is the beſt expedient for forming the minds of youth, by taking them off from the fruitleſs and airy exerciſes of the fancy, and rendering them ſerious, diligent and inquiſitive in the ſearch of knowledge and truth. Nature hath left every man a capacity of being agreeable, though not of ſhining in company. Thus we ſee how human nature is debaſed by the abuſe of that faculty, which is held the great diſtinction between men and brutes; and how little advantage many make of that, which might be the greateſt, the moſt laſting, and the moſt innocent as well as uſeful in life; in default of which they are forced to take up with dreſs and viſiting, or the more pernicious ones, of drink and vicious company. Thoſe who addict themſelves to ſuch a ſcandalous kind of life, ſcarce think of, or attend to any one thing beſides; for now-a-days 'tis too common to ſee licentiouſneſs paſs for humour, diſſimulation for honour, and vanity for every accompliſhment.

Before I take leave of the candid reader, I can't but take notice here, of a miſapprehenſion even ſome good Mathematicians make with regard to the right underſtanding ſome exponential Problems; as for inſtance, given the equation $x^{x^x} = y$, required the value of x when y is a *minimum?* Now when this Problem is examined, it will neither have *maximum* nor *minimum*. The quantity that admits of being a *minimum*, is not x^{x^x} but x^{xx}, for this will have a *minimum*. And the rule given in the enſuing Work is founded upon the ſame ſuppoſition, viz. that x^{x^x} is the ſame as x^{xx}, and $x^{x^{x^x}}$ the ſame as x^{xxx}, &c. for there is a great deal of difference between $\overline{x}|^{x^x}$ and $\overline{x^x}|^x$; let $x = 3$, then $\overline{3}|^{3^3}$ or 3^{27} far exceeds $\overline{3^3}|^3$ or 27^3, and when x^{xx} is a *minimum*, becauſe $\overline{x^x}|^x = x^{xx}$, the hyp. log. $x = -\frac{1}{2}$, and the number is .606, &c. but it will by no means anſwer, when $\overline{x}|^{x^x}$ is a *minimum*.

Alſo concerning the area of ſome particular curves, as for inſtance, given this equation of a curve $y^3 = ax^2 + bx^{\frac{1}{2}}y$, to find the area? Now by Mr. *Emerſon*'s rule, page 265 of his Fluxions, the area is $\frac{3}{5}yx + \frac{by^2}{5a\sqrt{x}}$ without any correction, which correction will always be different under different circumſtances, that is, according to the different points of the abſciſſa, that the area is ſuppoſed to begin at. Here if we ſuppoſe the area to begin with the abſciſſa, then in the vertex of the curve, both x, y, and the area are $= 0$, and the general or ſimple value of the area being $\frac{3}{5}yx + \frac{by^2}{5a\sqrt{x}}$, in the vertex it will be $0 + \frac{by^2}{5a\sqrt{x}}$; now to find the value of $\frac{byy}{5a\sqrt{x}}$ in the vertex, if we ſuppoſe y to be infinitely greater than x, in that point, then axx will vaniſh out of the equation of the curve, and we ſhall have $y^3 = by\sqrt{x}$, or $yy = b\sqrt{x}$, whence the value of $\frac{byy}{5a\sqrt{x}}$ in the vertex is $= \frac{bb}{5a}$, which taken out of the general expreſſion of the area, leaves the correct area $= \frac{3}{5}yx + \frac{by^2}{5a\sqrt{x}} - \frac{bb}{5a}$;

Again, ſuppoſe $yx^{\frac{1}{2}} = a + ay^3x^2$ be an equation to a curve, to find the area, ſome expreſſing the area one way and ſome another; now for a right underſtanding of this, we muſt obſerve, that ſuch curves have generally ſeveral parts or legs, and each ſeveral part gives a ſeveral area, and I imagine Mr. *Emerſon*'s method will give ſeveral expreſſions for the areas, by reducing the equation to ſix different forms, according to art. 5. p. 267, and then finding the area ſeparately for each form. Once more, let the relation between the abſciſſa and ordinate of a curve be expreſſed by $ay^nx^n = y^{2n+1} + x^{2n+1}$, the area will be $\frac{yx}{2} + \frac{ax^{n+1}}{\overline{4n+2}.y^n}$; or it may be $\frac{a\times\overline{n+1}}{2n+1} \times \frac{x^{n+1}}{y^n} - \frac{x^{2n+1}}{2y^{2n}}$ without correction;

and by page 264, of Mr. *Emerſon*'s Fluxions, the area is $= \frac{1}{2}xy - \frac{ay^{n+1}}{4n+2.x^n}$ without correction, which differs from both the foregoing; but theſe different expreſſions are all true, and I take them to ariſe partly from expunging ſome quantities to take in others from the equation of the curve, and partly from the ambiguity of the ordinate, which I ſhall now ſhew. As to the ambiguity of the ordinate,



Let ABC be the curve, whoſe equation is $x^3+y^3=axy$, and whoſe area is to be found; in theſe ſorts of curves, ſince there are the ſame powers of x as of y, ſo that writing x for y, and y for x in the equation of the curve, this equation ſhall not be changed, for this reaſon they return into themſelves, and are made up of two parts exactly ſimilar. Whence if $AD=x$, an ordinate erected at D will cut the curve twice in B and C, therefore y is either $=$ DB or DC, and from this it is indifferent as to the equation of the curve, whether the area ABD, or ACD be found; and hence comes that variety ariſing from the ambiguity of the ordinate.

And thus much I choos'd to ſay here by Way of Preface, rather than in the Book itſelf.

As my intention in publiſhing this plain and eaſy Introduction, is to bring Learners to the underſtanding of Fluxions, ſo likewiſe I hope it will give ſome eaſe to the Teacher, by furniſhing him with ſo great a variety of Examples; I ſhall ſubmit my endeavours to the judgment of the impartial reader, and hope he will not take amiſs, if any preſs errors ſhould inadvertently have crept in, conſidering my great diſtance from the preſs.

ERRATA.

Note, When *b* is ſet to the page, you muſt reckon from the bottom.

pag	line	read.
3	7	$\overline{a+b}\|^{5}$
8	12	$\overline{\frac{a}{x}}\|^{\frac{1}{n}}$
	13	$\frac{\overline{a^{n}}\|^{\frac{1}{n}}}{x}$
33	8	1.0.0 0.
	4b	$120a^{5}$
38	5	$\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.b^{4}}{1.\ 2.\ 3.\ 4.\ a^{n+4}}$
42	4	$\frac{n^{3}+3n^{2}+2n\times aaB^{3}}{6A^{n+3}}$
48	8	$\frac{b^{2}x^{2}}{2a}$
62	3	$\frac{6x^{2}}{a^{5}}\times\frac{x}{a}$
65	5b	$\overline{2rx}\|^{-\frac{1}{2}}$
70	6b	$\frac{8abc-a^{2}d-12b^{3}}{a^{10}}$
71	8	$\frac{7bb-4c}{8}$
86	13	Lines AB and CD
105	6b	$mx^{2}-tx+s$
110	laſt	$z^{x-1}\dot{x}$ log. a
161	7	$=\Phi$
	11	$\frac{2}{4.\ \overline{1.-1}}\Phi$, and $-\frac{1}{2}\Phi$
	12	$-a\Phi=$
197	9	$-\overline{v+n}.\ ccB$
207	6	$\frac{.01745329Dd}{2}$
209	7b	$\frac{R}{2r}$
218	6	$\dot{x}\sqrt{aa+2ax}$
234	laſt	Euclid. 1.
246	14	$=\dot{v}=0$
251	1b	$\frac{\sqrt{y^{2}\dot{x}^{2}+r\dot{y}^{2}}}{r}$.
255	ex.4	Fig. 26. in margin.
258	3b	$-\frac{z^{3}}{3}$
264	3b	$a^{2}\dot{t}\times\frac{1}{aa}-$
268	9	$\frac{d}{2}y^{2}$
	10	$\frac{d}{2}y^{2}\dot{y}$
	14	$=c$
	3b	$::t^{2}:c^{2}$
	laſt	$\dot{x}=\frac{-tty\dot{y}}{c^{2}x}$
270	12	$\frac{-\dot{x}x^{2}}{y}=-\frac{\dot{p}}{2}$
286	5	$\sqsubset 0$
288	8	$\dot{y}x=\frac{ax\dot{x}}{b}$
314	5b	is $\frac{2xx\sqrt{ax}}{5}$
321	2b	therefore by common Algebra
326	7b	velocity at B and A.
331	6	its Area
334	5	Fluent $\dot{y}x=\frac{dv^{m}x^{pn}}{pn}$
336	15	2140.30
	2b	$y=\overline{ax^{2}}\|^{\frac{1}{3}}$
339	2	$\frac{2\dot{y}}{a}\sqrt{a^{2}+4y^{2}}$.

THE

CONTENTS.

SECT. I.

OF Exponents or Indices and Powers, page 1
Multiplication of Indices, —— —— 3
Diviſion of Indices, —— —— 4
The Roots of Quantities expreſſed by Fractional Indices, 6

SECT. II.

Of Infinite Series by common Diviſion, —— 12
To find a Series expreſſing the Secant, —— 32
To find a Series expreſſing the Tangent, —— 33
To find a Series expreſſing the Co-tangent, — 35
Several univerſal Theorems for finding Series, — 37
Of reducing Surd Quantities into Infinite Series, 42
Sir *Iſaac Newton*'s Binomial Theorem, and its Uſe, 54
A Formula for the Power of an Infinitinomial, 69
Theorems for reverting of Series, — — 70

SECT. III.

Of FLUXIONS, Definitions, &c. —— — 73
PROP. I. To find the Fluxion of any ſimple variable Quantity, —— —— 92
PROP. II. To find the Fluxion of any given Power of any variable Quantity, — ib.

PROP.

PROP. III. To find the Fluxion of the Product of ſeveral variable Quantities, —— p. 93
PROP. IV. Given any Equation involving Fluents or flowing Quantities, to find the Fluxion, 94
PROP. V. To find the Fluxion of any Fraction, - 95
PROP. VI. To find the Fluxions of Powers where Surds are concerned, — —— 99
———— Several univerſal Theorems by which Quantities may be put into Fluxions by bare Subſtitution, — —
PROP. VII. To find the Fluxion of Logarithms, 107
PROP. VIII. To find the Fluxions of the Power of Logarithms, and of Exponentials, 108

OF SECOND FLUXIONS.

PROP. IX. Given an Equation containing the Fluxions of Quantities, to find the Fluents of any ſimple Fluxion, — — 115
PROP. X. To find the Fluent of any Power of a variable Quantity, multiplied by the Fluxion of the Root. —— — 116
PROP. XI. To find the Fluent of ſuch Fluxions, as involve two or more variable Quantities, 117
PROP. XII. To find the Fluent of any term when it is divided by the firſt Power of a variable Quantity, —— — 121
PROP. XIII. To find the Fluents where Surds are concerned, —— —— ib.
A Table of Fluxions and their Fluents, according to Mr. *Emerſon*'s Method, 128
To find Fluents from the Table, 129
To find Fluents by infinite Series, 130
PROP. XIV. XV. XVI. XVII. XVIII. XIX. XX. XXI. XXII. XXIII. are general for finding and comparing Fluents, —— —— 184
PROP. XXIV. To correct the Fluent of a given Fluxion, —— —— 216
PROP. XXV. To find the Logarithms, ——— 220

PROP.

PROP. XXVI. From a given hyperbolic Logarithm, to find the number anſwering thereto, 226
PROP. XXVII. To find a Series, which ſhall exhibit the Logarithms of the Sines, 227
PROP. XXVIII. To find a Series which ſhall exhibit the Logarithms of the Secants, 228
PROP. XXIX. To change the hyperbolic Logarithms into *Brigg's*, —— 229

SECT. IV.

PROB. I. To find the *Maxima* and *Minima* of Quantities, —— —— 230
A PROP. To ſolve all manner of ſimple Fluxionary Exponential Prob. — — 231
PROB. II. To draw Tangents to Curves, —— 248
PROB. III. To find the Length of Curve Lines, 251
PROB. IV. Given the Radius and Arch of a Circle, to find its Sine, —— — 260
PROB. V. Given the Radius and the Arch, to find the Secant, —— — 261

Of the CIRCULAR ARCH, its SINES, TANGENT and CHORD.

PROB. VI. Given the Radius and the verſed Sine, to find the Arch, —— 263
PROB. VII. Given the Radius and the Sine, to find the Arch, —— —— 264
PROB. VIII. Given the Radius and Tangent, to find the Arch, —— — ib.
PROB. IX. Given the Radius and Chord, to find the Arch, —— —— 265
PROB. X. Given the Radius and the Arch, to find the Tangent, —— —— ib.
SCHOLIUM. Having given the Radius and Arch, to find the Sine, Coſine, verſed Sine, Tangent, &c. —— — 266

PROB.

PROB. XI. To find the Area of Curves, —— p. 274
PROB. XII. A general Propoſition for finding the Area of all ſuch Figures as may be expreſſed in finite Terms, —— 285
PROB. XIII. To find the Surfaces of Solids generated by the Rotation of Planes about an Axis, —— —— 292
PROB. XIV. To find the Content of ſolid Bodies, 295
Solutions to a Collection of promiſcuous Problems, — —— —— 306

AN INTRODUCTION TO FLUXIONS.

SECT. I.

Of Exponents, or Indices of Quantities and Powers.

EXPONENT is a number placed over any involved quantity or power, to ſhew what height the root is raiſed; for powers, here, are nothing elſe but the products ariſing from the continual multiplication of a quantity into itſelf.

The powers raiſed from a binomial root $(a+b)$ have the ſign + to every term. And the powers raiſed from a reſidual root $(a-b)$ have their ſigns + and — interchangeably to every other term.

The exponents or indices of the powers of (a) the leading quantity continually decreaſe in arithmetical progreſſion, while the indices of (b) the other do continually increaſe in arithmetical progreſſion.

Thus in the 6th power it is $a^6, a^5, a^4, a^3, a^2, a$.
And $b, b^2, b^3, b^4, b^5, b^6$.

To involve $a+b$ to the 6th power without multiplying the quantities at length, this is the

RULE.

The power required muſt be made to conſiſt of one term more, than is the index of the power required, whereof the firſt and laſt are pure powers, the firſt of *a* and laſt of *b*, each involved to the power required, then all the intermediate terms are to be made up of both *a* and *b*, ſo that the ſum of their indices in any term is equal to the index of the required power.

And for finding the coefficients, or *unciæ*, of the ſeveral terms, the firſt is always 1, the ſecond is the index of the power required. And, in general, if the *uncia* of any term be multiplied by the index of the leading quantity (*a*) and divided by the number of terms to that place, it gives the *uncia* of the next following term.

Thus to find the 6th power of $a + b$, the terms without the *unciæ* will be a^6, a^5b, a^4b^2, a^3b^3, a^2b^4, ab^5, b^6; and the particular *unciæ* or coefficients, by the Rule are 1, 6, $\frac{6 \times 5}{2} = 15$, $\frac{15 \times 4}{3} = 20$, $\frac{20 \times 3}{4} = 15$, $\frac{15 \times 2}{5} = 6$, and $\frac{6 \times 1}{6} = 1$.

Therefore $\overline{a+b}|^{6} = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; and $\overline{a-b}|^{6} = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

Sir Iſaac Newton diſcovered a theorem to expreſs any binomial $(a + b)$ or reſidual $(a - b)$ in ſimple terms.

Let m be any power at pleaſure, then the mth power of the binominal $a + b$ will be $\overline{a+b}|^{m} =$

$$a^m + ma^{m-1}b + \frac{m.m-1}{1.2} \times a^{m-2}b^2 + \frac{m.m-1.m-2}{1.2.3} \times a^{n-3}b^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4} \times a^{m-4}b^4 +$$

$m.m - 1$

$$\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}}{1.2.3.4.5} \times a^{m-5}b^5 + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}.\overline{m-5}}{1.2.3.4.5.6} \times a^{m-6}b^6 \text{ \&c.}$$

By this ſeries the terms of any power may very readily be found by ſubſtituting the index of the given power for m.

Thus to raiſe $\overline{a+b}$ to the 5th power; then $\overline{a+b}|^{m} = \overline{a = b}|^{5}$. Here $m = 5$.

1. $a^m =$ — — a^5
2. $ma^{m-1}b =$ — — $5a^4b$
3. $\frac{m.\overline{m-1}}{1.2} a^{m-2}b^2 =$ = $10a^3b^2$
4. $\frac{m.\overline{m-1}.\overline{m-2}}{1.2.3} a^{m-3}b^3 =$ — $10a^2b^3$
5. $\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{1.2.3.4} a^{m-4}b^4 =$ — $5ab^4$
6. $\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}}{1.2.3.4.5} a^{m-5}b^5 =$ — b^5

Therefore $\overline{a+b}|^{5} = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

1. Whence the index of x is — = x^1
2. of xx is = x^2
3. of xxx is = x^3
4. of $xxxx$ is = x^4
5. of $xxxxx$ is = x^5
&c. &c.

Multiplication of Indices.

The ſum of the indices expreſſes the power; and negative powers, as well as affirmative, are multiplied by adding their indices or exponents.

EXAMPLES.

1. $x \times x^2 = x^{1+2} = x^3$
2. $x^2 \times x^3 = x^{2+3} = x^5$
3. $x^3 \times x^4 = x^{3+4} = x^7$
4. $x^3 \times x^5 = x^{3+5} = x^8$
5. $x^2 \times x^{-3} = x^{2-3} = x^{-1}$.

In general, if we put m and n for the exponents of any two quantities, then x^m expreſſes any power of x, as before.

1. $x^m \times x^n = x^{m+n}$
2. $x^m \times x^{-m} = x^{m-m} = x^0 = 1$
3. $x^{-m} \times x^n = x^{-m+n}$
4. $x^{-m} \times x^{-n} = x^{-m-n}$.

Alſo x^m raiſed to the n^{th} power is $= x^{mn}$.
And x^{-m} raiſed to the n^{th} power is $= x^{-mn}$.

Diviſion of Indices.

The difference of the indices expreſſes the power; and negative powers as well as affirmative are divided by ſubtracting their indices.

EXAMPLES.

1. $\dfrac{x}{x} = x^{1-1} = x^0 = 1$
2. $\dfrac{x^2}{x} = x^{2-1} = x$
3. $\dfrac{x^3}{x^2} = x^{3-2} = x$
4. $\dfrac{x^{-5}}{x^{-2}} = x^{-5+2} = x^{-3} = \dfrac{1}{x^3}$.
5. $\dfrac{x^{-2}}{x^{-5}} = x^{-2+5} = x^3 = \dfrac{1}{x^{-3}}$.
6. $\dfrac{2}{x^3} = 2x^{-3}$.

In general, 1. $\frac{x^m}{x^n} = x^{m-n}$

2. $\frac{x^{-m}}{x^n} = x^{-m-n}$

3. $\frac{x^n}{x^{-m}} = x^{n+m}$

4. $\frac{x^{-m}}{x^{-n}} = x^{-m+n}$

If $\frac{z}{a} = \frac{a^m}{x^m}$, then $z = \frac{a^{m+1}}{x^m}$.

RULE.

When any quantity is in the denominator of a fraction, put that quantity in the numerator, and change the ſign of the exponent or index, and the expreſſion is put in another form of the ſame value. See Mr. *Emerſon*'s Algebra, p. 50.

Thus, 1. $\frac{1}{x}$ is $= x^{-1}$. For $\frac{1}{x} = \frac{x}{xx} = \frac{x^1}{x^2} = x^{1-2} = x^{-1}$

2. $\frac{1}{x^2} = x^{-2}$

3. $\frac{1}{x^3} = x^{-3}$

4. $\frac{1}{x^{-3}} = x^3$

5. $\frac{1}{x^4} = x^{-4}$

6. $\frac{1}{a^{3+n}} = a^{-3-n}$

7. $\frac{\frac{1}{2}}{121} = \frac{1}{2 \times 121} = 242^{-1}$

8. $\frac{1}{aa + xx} = \overline{aa + xx}|^{-1}$

9. $\dfrac{1}{\overline{aa+xx}|^2} = \overline{aa+xx}|^{-2}$

10. $\dfrac{1}{\overline{aa+xx}|^3} = \overline{aa+xx}|^{-3}$

11. $\dfrac{1}{\overline{aa+xx}|^4} = \overline{aa+xx}|^{-4}$

Alſo 12. $\dfrac{aab}{a+x}$ may be expreſſed $= aab \times \overline{a+x}|^{-1}$

13. $\dfrac{aab}{\overline{a+x}|^2} = aab \times \overline{a+x}|^{-2}$

14. $\dfrac{aab}{\overline{a+x}|^3} = aab \times \overline{a+x}|^{-3}$

15. $\dfrac{aab}{\overline{a+x}|^m} = aab \times \overline{a+x}|^{-m}$

16. $\dfrac{bc}{x^2y^3} = bcx^{-2}y^{-3}$

17. $\dfrac{x^2}{x^3} = x^2x^{-3} = \dfrac{1}{x^{-2}x^3} = x^{-3+2} = x^{-1}$

18. $\dfrac{xy}{y - \frac{x^3}{a^3}z} = \dfrac{a^3xy}{a^3y - x^3z} = a^3xy \times \overline{a^3y - x^3z}|^{-1}$,

The nth root of x^m is $= x^{\frac{m}{n}}$

The nth root of $x^{-m} = x^{\frac{-m}{n}}$

Alſo $x^{\frac{1}{m}}$ raiſed to the mth power is $= x^{\frac{1}{m}\times m} = x^{\frac{m}{m}} = x$

And $x^{\frac{1}{m}}$ raiſed to the nth power is $= x^{\frac{1}{m}\times n} = x^{\frac{n}{m}}$

Alſo $\overline{\dfrac{a}{x}}|^m = \dfrac{a^m}{x^m} = \sqrt[\frac{1}{m}]{\dfrac{a}{x}} = \dfrac{\sqrt[\frac{1}{m}]{a}}{\sqrt[\frac{1}{m}]{x}}$.

The roots of quantities may be expreſſed by fractional indices, thus.

RULE.

Divide the index of the quantity by the number expreſſing the root, the quotient is the index of the root required. See Mr. *Emerſon*'s Algebra, p. 48.

Thus, 1. $\sqrt{x} = x^{\frac{1}{2}}$. 2. $\sqrt[3]{x} = x^{\frac{1}{3}}$. And, 3. $\sqrt[5]{x} = x^{\frac{1}{5}}$.

Alſo, 4. $\sqrt{3ax^2} = \overline{3ax^2}|^{\frac{1}{2}} = a^{\frac{1}{2}}x\sqrt{3}$.

5. $\sqrt[3]{3ax^2} = \overline{3ax^2}|^{\frac{1}{3}} = a^{\frac{1}{3}}x^{\frac{2}{3}}\sqrt[3]{3}$.

6. $\sqrt[4]{3ax^2} = \overline{3ax^2}|^{\frac{1}{4}} = a^{\frac{1}{4}}x^{\frac{1}{2}}\sqrt[4]{3}$.

7. $\sqrt[5]{3ax^2} = \overline{3ax^2}|^{\frac{1}{5}} = a^{\frac{1}{5}}x^{\frac{2}{5}}\sqrt[5]{3}$.

Alſo, 8. $\sqrt{x^3}$ or $\overline{x^3}|^{\frac{1}{2}} = x^{\frac{3}{2}}$

9. $\sqrt[3]{x^3} = x^{\frac{3}{3}} = x$

10. $\sqrt[4]{x^3} = x^{\frac{3}{4}}$

11. $\sqrt[5]{x^3} = x^{\frac{3}{5}}$

12. $\sqrt{\frac{1}{x}} = \frac{1}{x^{\frac{1}{2}}}$

13. $\sqrt[3]{\frac{1}{x}} = \frac{1}{x^{\frac{1}{3}}}$

14. $\sqrt[4]{\frac{1}{x}} = \frac{1}{x^{\frac{1}{4}}}$

Again, 1. $\frac{1}{\sqrt[3]{x^4}} = \frac{1}{x^{\frac{4}{3}}} = x^{-\frac{4}{3}}$

2. $\frac{1}{\sqrt[3]{x^3}} = \frac{1}{x} = x^{-1}$

3. $\frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$

4. $\frac{1}{\sqrt[3]{x}} = \frac{1}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}}$

5. $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

6. $\sqrt[3]{x^4} = x^{\frac{4}{3}}$

7. $\sqrt{x^5} = x^{\frac{5}{2}}$

8. $\dfrac{1}{\sqrt{aa-xx}} = \overline{aa-xx}^{\,-\frac{1}{2}}$

9. $\sqrt[3]{\dfrac{aa}{aa+xx}} = a^{\frac{2}{3}} \times \overline{aa+xx}^{\,-\frac{1}{3}}$

If $\frac{1}{m}$ and $\frac{1}{n}$ denote the *m*th roots, and *n*th roots of any quantities generally; they may be multiplied and divided by the foregoing rules, as follows.

EXAM.

1. $x^{\frac{1}{m}} \times y^{\frac{1}{m}} = \overline{xy}^{\frac{1}{m}}$

2. $\overline{ab}|^{\frac{1}{m}} \times \overline{cx}|^{\frac{1}{m}} = \overline{abcx}|^{\frac{1}{m}}$

3. $x^{\frac{1}{m}} \times a = \overline{a^m x}|^{\frac{1}{m}}$

4. $a^{\frac{1}{n}} \times x^{\frac{1}{m}} = \overline{a^m x^n}|^{\frac{1}{nm}}$

Alſo 1. $a^{\frac{1}{n}}$ divided by $x^{\frac{1}{n}} = \dfrac{a^{\frac{1}{n}}}{x^{\frac{1}{n}}} = \overline{\dfrac{a}{x}}|^{\frac{1}{n}}$

2. a divided by $x^{\frac{1}{n}} = \dfrac{a}{x^{\frac{1}{n}}} = \overline{\dfrac{a^n}{x}}|^{\frac{1}{n}}$

3. $a^{\frac{1}{n}}$ divided by $x^{\frac{1}{m}} = \dfrac{a^{\frac{1}{n}}}{x^{\frac{1}{m}}} = \overline{\dfrac{a^m}{x^n}}|^{\frac{1}{nm}}$

If the powers be the ſame, then the power affected by the ſum or difference of the exponents will be the product or quotient.

EXAM.

EXAM.

Thus 1. $x^{\frac{n}{m}} \times x^{\frac{c}{p}}$ is $= x^{\frac{np+mc}{mp}}$. And the $\frac{n}{m}$ power of $x^{\frac{c}{p}}$ is $= x^{\frac{nc}{mp}}$.

2. $\frac{1}{x} \times \frac{1}{\overline{x^n}|^{\frac{1}{m}}}$, or $x^{-1} \times x^{-\frac{n}{m}} = x^{\frac{-\overline{m+n}}{m}} =$

$$\frac{1}{\overline{x^{m+n}}|^{\frac{1}{m}}}.$$

3. $\frac{1}{x^{\frac{n}{m}}} \times \frac{1}{x^{\frac{c}{p}}}$, or $x^{-\frac{n}{m}} \times x^{-\frac{c}{p}} = x^{\frac{-\overline{pn+cm}}{pm}}$

$$= \frac{1}{x^{\frac{pn+cm}{pm}}} = \frac{1}{\overline{x^{pn+cm}}|^{\frac{1}{pm}}}.$$

4. $\frac{1}{\overline{ax^m}|^{\frac{1}{n}}} \times \frac{1}{\overline{x^r}|^{\frac{1}{s}}}$, or $a^{-\frac{m}{n}} \times x^{-\frac{m}{n}} \times x^{-\frac{r}{s}}$

will be $= a^{-\frac{m}{n}} \times x^{\frac{-sm-rn}{sn}} = \frac{1}{a^{\frac{m}{n}} \times x^{\frac{sm+rn}{sn}}}$, or

$$\frac{1}{\overline{a^m}|^{\frac{1}{n}} \times \overline{x^{sm+rn}}|^{\frac{1}{sn}}}.$$

Cor.

Thus $x^{\frac{1}{2}} \times x^{\frac{5}{2}} = x^{\frac{1}{2}+\frac{5}{2}} = x^{\frac{6}{2}} = x^3$.
And $x^{\frac{2}{3}} \times x^{\frac{3}{4}} = x^{\frac{2}{3}+\frac{3}{4}} = x^{\frac{17}{12}}$.

In

In Diviſion.

1. $x^{\frac{n}{m}}$ divided by $x^{\frac{c}{p}} = x^{\frac{pn-cm}{pm}}$.

2. $x^{-\frac{1}{n}}$ divided by $x^{\frac{1}{m}} = x^{-\frac{1}{n}-\frac{1}{m}} = x^{\frac{\overline{-m+n}}{mn}}$;

3. $\dfrac{1}{\overline{x^n}|^{\frac{1}{m}}}$ divided by $\dfrac{1}{x}$, or $x^{-\frac{n}{m}}$ divided by x^{-1}

$$= x^{\frac{-n+m}{m}} = \frac{1}{\overline{x^{n+m}}|^{\frac{1}{m}}};$$

By changing the ſign of the index, tranſpoſes the denominator into the numerator; and, on the contrary, the numerator into the denominator.

EXAM.

1. $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$ or $\dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{1}{6}}$

2. $\dfrac{\sqrt{x}}{\sqrt[4]{x}} = \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{4}}} = x^{\frac{1}{2}-\frac{1}{4}} = x^{\frac{2}{8}} = x^{\frac{1}{4}}$

3. $\dfrac{\sqrt{x}}{\sqrt[5]{x}} = \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{5}}} = x^{\frac{1}{2}-\frac{1}{5}} = x^{\frac{3}{10}}$

4. $\dfrac{\sqrt[3]{x}}{\sqrt[4]{x}} = \dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{4}}} = x^{\frac{1}{3}-\frac{1}{4}} = x^{\frac{1}{12}}$

5. $\dfrac{\sqrt[3]{x}}{\sqrt[5]{x}} = \dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{5}}} = x^{\frac{1}{3}-\frac{1}{5}} = x^{\frac{2}{15}}$

6. $\dfrac{x^{\frac{7}{2}}}{x^{\frac{3}{2}}} = x^{\frac{7-3}{2}} = x^{\frac{4}{2}} = x^2$.

Corollary.

In like manner may they be involved, and the root extracted.

Thus. The ſquare of $x^{\frac{1}{2}}$ is $= x^{\frac{1}{2}\times 2} = x$

The ſquare of $x^{\frac{3}{2}} = x^{\frac{3}{2}\times 2} = x^{3}$

The ſquare root of $x^{\frac{2}{3}}$ is $x^{\frac{2}{3}\div 2} = x^{\frac{2}{3\times 2}} = x^{\frac{1}{3}}$

The cube root of $x^{\frac{3}{4}}$ is $x^{\frac{3}{4}\div 3} = x^{\frac{3}{4\times 3}} = x^{\frac{1}{4}}$.

SECT.

SECT. II.

Of Infinite Series.

ANY fractional quantity, whoſe denominator is a compound one, may be put into an infinite ſeries by diviſion; and ſo alſo may a ſurd, by extracting the root of it in the manner of a decimal fraction, the root thereof is an infinite ſeries, and the law of continuation in many caſes is evident, by getting a few firſt terms.

EXAMPLE I.

Find the value of $\dfrac{\frac{1}{2}}{1+\frac{1}{2}}$ *in an infinite ſeries.*

$1+\frac{1}{2})\frac{1}{2}+0(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$ &c. *ad infinitum*, where the law of continuation is evident.

$$\begin{array}{l}
\underline{\frac{1}{2}+\frac{1}{4}} \\
\quad -\frac{1}{4}+0 \\
\quad \underline{-\frac{1}{4}-\frac{1}{8}} \\
\qquad +\frac{1}{8}+0 \\
\qquad \underline{+\frac{1}{8}+\frac{1}{16}} \\
\qquad\quad -\frac{1}{16}+0 \\
\qquad\quad \underline{-\frac{1}{16}-\frac{1}{32}} \\
\qquad\qquad +\frac{1}{32}+0 \\
\qquad\qquad \underline{+\frac{1}{32}+\frac{1}{64}} \\
\qquad\qquad\quad \underline{-\frac{1}{64}\ \text{\&c.}}
\end{array}$$

Therefore $\dfrac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$ &c.

EXAM.

EXAM. 2.

Find the value of $\frac{x}{1+x}$ *in an infinite series.*

$1+x)x+0(x-x^2+x^3-x^4+x^5-$&c. where the law of continuation is evident.

$$\begin{array}{l}
x+xx \\
\hline
-xx+0 \\
-xx-x^3 \\
\hline
\qquad +x^3+0 \\
\qquad +x^3+x^4 \\
\hline
\qquad\qquad -x^4+0 \\
\qquad\qquad -x^4-x^5 \\
\hline
\qquad\qquad\qquad +x^5+0 \\
\qquad\qquad\qquad +x^5+x^6 \\
\hline
\qquad\qquad\qquad\qquad -x^6 \text{ \&c.} \\
\hline
\end{array}$$

Therefore $\frac{x}{1+x}=x-x^2+x^3-x^4+x^5-x^6$ &c.

EXAM.

EXAM. 3.

Let it be required to find the value of $\frac{1}{a-x}$ *in an infinite series.*

$a-x)\,1+0\left(\frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\frac{x^4}{a^5}\right.$ &c. the continuation is evident.

$$\begin{array}{l}
1-\frac{x}{a} \\
\hline
+\frac{x}{a}+0 \\
+\frac{x}{a}-\frac{xx}{aa} \\
\hline
+\frac{xx}{aa}+0 \\
+\frac{xx}{aa}-\frac{x^3}{a^3} \\
\hline
+\frac{x^3}{a^3}+0 \\
+\frac{x^3}{a^3}-\frac{x^4}{a^4} \\
\hline
+\frac{x^4}{a^4}+0 \\
+\frac{x^4}{a^4}-\frac{x^5}{a^5} \\
\hline
+\frac{x^5}{a^5}\ \text{\&c.} \\
\hline
\end{array}$$

Proof.

1. $\frac{a-x}{1}\times\frac{1}{a}=\frac{a-x}{a}=1-\frac{x}{a}$

2. $\frac{a-x}{1}\times\frac{x}{a^2}=\frac{ax-xx}{a^2}=\frac{x}{a}-\frac{xx}{a^2}$

$$3.\quad \frac{a-x}{1}\times\frac{x^2}{a^3}=\frac{ax^2-x^3}{a^3}=\frac{x^2}{a^2}-\frac{x^3}{a^3}$$

$$4.\quad \frac{a-x}{1}\times\frac{x^3}{a^4}=\frac{ax^3-x^4}{a^4}=\frac{x^3}{a^3}-\frac{x^4}{a^4}$$

$$5.\quad \frac{a-x}{1}\times\frac{x^4}{a^5}=\frac{ax^4-x^5}{a^5}=\frac{x^4}{a^4}-\frac{x^5}{a^5}$$

Therefore $\frac{1}{a-x}=\frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\frac{x^4}{a^5}$ &c. the ſeries required.

Exam. 4.

To expreſs $\frac{1}{a-2x}$ *in a ſeries.*

$$(a-2x)\,1+0\left(\frac{1}{a}+\frac{2x}{aa}+\frac{4x^2}{a^3}+\frac{8x^3}{a^4}+\frac{16x^4}{a^5}\ \&c.\right.$$

$$\begin{array}{llll}
\underline{1-\frac{2x}{a}} & & & \\
+\frac{2x}{a}+0 & & & \\
\underline{+\frac{2x}{a}-\frac{4x^2}{a^2}} & & & \\
 & +\frac{4x^2}{a^2}+0 & & \\
 & \underline{+\frac{4x^2}{a^2}-\frac{8x^3}{a^3}} & & \\
 & & +\frac{8x^3}{a^3}+0 & \\
 & & \underline{+\frac{8x^3}{a^3}-\frac{16x^4}{a^4}} & \\
 & & & \&c.
\end{array}$$

Therefore $\frac{1}{a-2x}=\frac{1}{a}$ &c.

Exam.

EXAM. 5.

To find the value of $\frac{1}{a-3x}$ *in an infinite series.*

$$a-3x)\,1+0\left(\frac{1}{a}+\frac{3x}{a^2}+\frac{9x^2}{a^3}+\frac{27x^3}{a^4}+\frac{81x^4}{a^5}\ \&c.\right.$$

$$\underline{1-\frac{3x}{a}}$$

$$+\frac{3x}{a}+0$$

$$\underline{+\frac{3x}{a}-\frac{9x^2}{a^2}}$$

$$+\frac{9x^2}{a^2}+0$$

$$\underline{+\frac{9x^2}{a^2}-\frac{27x^3}{a^3}}$$

$$+\frac{27x^3}{a^3}+0$$

$$\underline{+\frac{27x^3}{a^3}-\frac{81x^4}{a^4}}$$

$$+\frac{81x^4}{a^4}+0$$

$$\underline{+\frac{81x^4}{a^4}-\frac{243x^5}{a^5}}$$

$$\underline{+\frac{243x^5}{a^5}\ \&c.}$$

Proof.

1. $\frac{a-3x}{1}\times\frac{1}{a}=\frac{a-3x}{a}=1-\frac{3x}{a}$

2. $\frac{a-3x}{1}\times\frac{3x}{a^2}=\frac{3ax-9x^2}{aa}=\frac{3x}{a}-\frac{9x^2}{aa}$

3. $\frac{a-3x}{1}\times\frac{9x^2}{a^3}=\frac{9ax^2-27x^3}{a^3}=\frac{9x^2}{aa}-\frac{27x^3}{a^3}$

4. $a-$

4. $\frac{a-3x}{1}\times\frac{27x^3}{a^4}=\frac{27ax^3-81x^4}{a^4}=\frac{27x^3}{a^3}-\frac{81x^4}{a^4}$

5. $\frac{a-3x}{1}\times\frac{81x^4}{a^5}=\frac{81ax^4-243x^5}{a^5}=\frac{81x^4}{a^4}-\frac{243x^5}{a^5}$

Therefore $\frac{1}{a-3x}=\frac{1}{a}+\frac{3x}{aa}+\frac{9x^2}{a^3}+\frac{27x^3}{a^4}+\frac{81x^4}{a^5}$ &c. the ſeries required.

Exam. 6.

Let $\frac{zz}{x+y}$ *be given to find the value in an infinite ſeries.*

$$
\begin{array}{l}
x+y)\,zz+0\left(\frac{zz}{x}-\frac{z^2y}{x^2}+\frac{z^2y^2}{x^3}-\frac{z^2y^3}{x^4}\ \&c.\right.\\
\quad zz+\frac{zzy}{x}\\
\hline
\quad\quad -\frac{zzy}{x}+0\\
\quad\quad -\frac{zzy}{x}-\frac{z^2y^2}{x^2}\\
\hline
\quad\quad\quad +\frac{z^2y^2}{x^2}+0\\
\quad\quad\quad +\frac{z^2y^2}{x^2}+\frac{z^2y^3}{x^3}\\
\hline
\quad\quad\quad\quad -\frac{z^2y^3}{x^3}+0\\
\quad\quad\quad\quad -\frac{z^2y^3}{x^3}-\frac{z^2y^4}{x^4}\\
\hline
\quad\quad\quad\quad\quad +\frac{z^2y^4}{x^4}\ \&c.
\end{array}
$$

Proof.

1. $$\frac{x+y}{1} \times \frac{zz}{x} = \frac{z^2x + z^2y}{x} = z^2 + \frac{z^2y}{x}$$

2. $$\frac{x+y}{1} \times \frac{-z^2y}{x^2} = \frac{-z^2xy - z^2y^2}{x^2} = \frac{-z^2y}{x} - \frac{z^2y^2}{x^2}$$

3. $$\frac{x+y}{1} \times \frac{z^2y^2}{x^3} = \frac{z^2xy^2 + z^2y^3}{x^3} = \frac{z^2y^2}{x^2} + \frac{z^2y^3}{x^3}$$

4. $$\frac{x+y}{1} \times \frac{-z^2y^3}{x^4} = \frac{-xz^2y^3 - z^2y^4}{x^4} = \frac{-z^2y^3}{x^3} - \frac{z^2y^4}{x^4}$$

EXAM. 7.

Let $\frac{1}{1+xx}$ *be put into a ſeries.*

$$\begin{array}{l}
1+xx)\,1 \quad (1 - x^2 + x^4 - x^6 + x^8 \ \&c. \\
\qquad\quad \underline{1 + xx} \\
\qquad\qquad -xx + 0 \\
\qquad\qquad \underline{-xx - x^4} \\
\qquad\qquad\qquad +x^4 + 0 \\
\qquad\qquad\qquad \underline{+x^4 + x^6} \\
\qquad\qquad\qquad\qquad -x^6 + 0 \\
\qquad\qquad\qquad\qquad \underline{-x^6 - x^8} \\
\qquad\qquad\qquad\qquad\qquad \underline{+x^8 \ \&c.}
\end{array}$$

EXAM.

EXAM. 8.

Find the value of $\frac{aa+xx}{a+x}$ *in an infinite ſeries.*

$$\begin{array}{l}
a+x)\,aa+xx\,(a-x+\frac{2x^2}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}\ \&c. \\
\quad aa+ax \\ \hline
\quad -ax+xx \\
\quad -ax-xx \\ \hline
\qquad +2xx+0 \\
\qquad +2xx+\frac{2x^3}{a} \\ \hline
\qquad\quad -\frac{2x^3}{a}+0 \\
\qquad\quad -\frac{2x^3}{a}-\frac{2x^4}{a^2} \\ \hline
\qquad\qquad +\frac{2x^4}{a^2}+0 \\
\qquad\qquad +\frac{2x^4}{a^2}+\frac{2x^5}{a^3}\ \&c. \\ \hline
\end{array}$$

Therefore $\frac{aa+xx}{a+x}=a-x+\frac{2x^2}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}$ &c.

EXAM. 9.

Let it be required to put $\frac{1}{xx+1}$ *into a ſeries.*

$$xx+1)\,1\,\Big(\frac{1}{xx}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}\ \text{\&c.}$$

$$1+\frac{1}{xx}$$

$$\overline{\qquad\qquad}$$

$$-\frac{1}{xx}+0$$

$$-\frac{1}{xx}-\frac{1}{x^4}$$

$$\overline{\qquad\qquad\qquad}$$

$$+\frac{1}{x^4}+0\ \text{\&c.}$$

Proof.

$$1.\ \frac{xx+1}{1}\times\frac{1}{xx}=\frac{xx+1}{xx}=1+\frac{1}{xx}$$

$$2.\ \frac{xx+1}{1}\times-\frac{1}{x^4}=\frac{-xx-1}{x^4}=-\frac{1}{xx}-\frac{1}{x^4}$$

&c. &c.

Therefore $\frac{1}{xx+1}=\frac{1}{xx}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}+$ &c.

Corollary.

Hence $\frac{1}{1-xx}=1+x^2+x^4+x^6+x^8$ &c. by the ſame way.

EXAM. 10.

Let the value of $\frac{x^2y}{x^2-y^2}$ *be found in an infinite ſeries.*

$$x^2-y^2)\,x^2y\,(y+\frac{y^3}{x^2}+\frac{y^5}{x^4}+\frac{y^7}{x^6}+\frac{y^9}{x^8}\ \text{\&c.}$$

$$\underline{x^2y-y^3}$$

$$+y^3+0$$

$$\underline{+y^3-\frac{y^5}{x^2}}$$

$$+\frac{y^5}{x^2}+0$$

$$\underline{+\frac{y^5}{x^2}-\frac{y^7}{x^4}}$$

$$+\frac{y^7}{x^4}\ \text{\&c.}$$

EXAM. 11.

Find the value of $\frac{x^3z}{x^2-y^2}$ *in an infinite ſeries.*

$$x^2-y^2)\,x^3z\,(xz+\frac{y^2z}{x}+\frac{y^4z}{x^3}+\frac{y^6z}{x^5}\ \text{\&c.}$$

$$\underline{x^3z-xzy^2}$$

$$+xzy^2+0$$

$$\underline{+xzy^2-\frac{y^4z}{x}}$$

$$+\frac{y^4z}{x}+0$$

$$\underline{+\frac{y^4z}{x}-\frac{y^6z}{x^3}}$$

$$+\frac{y^6z}{x^3}+0$$

$$\underline{+\frac{y^6z}{x^3}-\frac{y^8z}{x^5}}$$

$$+\frac{y^8z}{x^5}\ \text{\&c.}$$

This may be proved as follows.

$$\text{For } xz + \frac{y^2z}{x} + \frac{y^4z}{x^3} + \frac{y^6z}{x^5} \text{ \&c.}$$

$$\times \text{ by } x^2 - y^2$$

$$\begin{array}{r} x^3z + xy^2z + \frac{y^4z}{x} + \frac{y^6z}{x^3} \text{ \&c.} \\ - xy^2z - \frac{y^4z}{x} - \frac{y^6z}{x^3} \text{ \&c.} \\ \hline \end{array}$$

$$\text{Produces } x^3z \;.\; 0 \;.\; 0 \;.\; 0$$

EXAM.

EXAM. 12.

Reduce $\frac{1}{6+x}$ *into a ſeries.*

$$6+x)\,1\,\left(\frac{1}{6}-\frac{x}{36}+\frac{x^2}{216}-\frac{x^3}{1296}+\frac{x^4}{7776}\text{ \&c.}\right.$$

$$\underline{1+\frac{x}{6}}$$

$$-\frac{x}{6}+0$$

$$\underline{-\frac{x}{6}-\frac{x^2}{36}}$$

$$+\frac{x^2}{36}+0$$

$$\underline{+\frac{x^2}{36}+\frac{x^3}{216}}$$

$$-\frac{x^3}{216}+0$$

$$\underline{-\frac{x^3}{216}-\frac{x^4}{1296}}$$

$$+\frac{x^4}{1296}+0$$

$$\underline{+\frac{x^4}{1296}+\frac{x^5}{7776}}$$

$$\underline{-\text{ \&c.}}$$

Therefore $\frac{1}{6+x}=\frac{1}{6}-\frac{x}{36}+\frac{x^2}{216}-\frac{x^3}{1296}+\frac{x^4}{7776}$ &c.

Corollary.

Hence $\frac{x}{6+x}=\frac{x}{6}-\frac{x^2}{36}+\frac{x^3}{216}-\frac{x^4}{1296}$ $+\frac{x^5}{7776}$ &c.

EXAM. 13.

Reduce $\frac{1}{1-3x+xx}$ *into a ſeries.*

$$\begin{array}{l}
1-3x+xx)1(1+3x+8x^2+21x^3+55x^4+144x^5+\text{ \&c.}\\
\underline{1-3x+xx}\\
+3x-xx+0\\
\underline{+3x-9xx+3x^3}\\
+8x^2-3x^3+0\\
\underline{+8x^2-24x^3+8x^4}\\
+21x^3-8x^4+0\\
\underline{+21x^3-63x^4+21x^5}\\
+55x^4-21x^5+0\\
\underline{+55x^4-165x^5+55x^6}\\
+144x^5-55x^6+0\\
\underline{+144x^5-432x^6+144x^7}\\
\underline{+377x^6-144x^7}\\
\underline{\text{\&c.}}
\end{array}$$

Therefore $\frac{1}{1-3x+xx}=1+3x+8x^2+21x^3$ $+55x^4+144x^5+$ &c.

EXAM.

EXAM. 14.

Reduce $\dfrac{1-x}{\overline{1+x}|^{3}}$ *into an infinite ſeries.*

$$1+3x+3x^2+x^3)\,1-x\,(1-4x+9x^2-16x^3+25x^4-36x^5+49x^6\ \&c.$$

$$\begin{array}{l}
1+3x+3x^2+x^3 \\
\hline
-4x-3x^2-x^3 \\
-4x-12x^2-12x^3-4x^4 \\
\hline
+9x^2+11x^3+4x^4 \\
+9x^2+27x^3+27x^4+9x^5 \\
\hline
-16x^3-23x^4-9x^5 \\
-16x^3-48x^4-48x^5-16x^6 \\
\hline
+25x^4+39x^5+16x^6 \\
+25x^4+75x^5+75x^6\ \&c. \\
\hline
-36x^5-59x^6\ \&c. \\
-36x^5-108x^6\ \&c. \\
\hline
\&c. \\
\hline
\end{array}$$

Therefore $\dfrac{1-x}{\overline{1+x}|^{3}} = 1-4x+9x^2-16x^3+25x^4$
$-36x^5+49x^6$ &c.

EXAM.

EXAM. 15.

Put $\frac{1+6x^2+x^4}{\overline{1-xx}|^3}$ *into an infinite ſeries.*

$$1-3x^2+3x^4-x^6)\,1+6x^2+x^4\,(1+9x^2+25x^4+49x^6+81x^8 \text{ \&c.}$$

$$1-3x^2+3x^4-x^6$$

$$+9x^2-2x^4+x^6$$

$$+9x^2-27x^4+27x^6-9x^8$$

$$+25x^4-26x^6+9x^8$$

$$+25x^4-75x^6+75x^8 \text{ \&c.}$$

$$+49x^6-66x^8 \text{ \&c.}$$

$$+49x^6-147x^8 \text{ \&c.}$$

$$+81x^8 \text{ \&c.}$$

$$+81x^8 \text{ \&c.}$$

Therefore $\frac{1+6x^2+x^4}{\overline{1-xx}|^3} = 1+9x^2+25x^4+49x^6+81x^8$ &c.

EXAM. 16.

Put $\frac{-6x}{xx-2x-1}$ *into an infinite ſeries.*

$$xx-2x-1)-6x\left(-\frac{6}{x}-\frac{12}{xx}-\frac{30}{x^3}-\frac{72}{x^4}\text{ \&c.}\right.$$

$$\underline{-6x+12+\frac{6}{x}}$$

$$-12-\frac{6}{x}$$

$$\underline{-12+\frac{24}{x}+\frac{12}{x^2}}$$

$$-\frac{30}{x}-\frac{12}{x^2}$$

$$\underline{-\frac{30}{x}+\frac{60}{x^2}+\frac{30}{x^3}}$$

$$-\frac{72}{x^2}-\frac{30}{x^3}$$

$$\underline{-\frac{72}{x^2}+\frac{144}{x^3}}$$

$$\underline{-\frac{174}{x^3}\text{ \&c.}}$$

Therefore $\frac{-6x}{xx-2x-1}=-\frac{6}{x}-\frac{12}{xx}-\frac{30}{x^3}$ $-\frac{72}{x^4}$ &c.

EXAM.

EXAM. 17.

Put $\frac{a^3xy}{a^3y+zx^3}$ *into an infinite ſeries.*

$$a^3y+zx^3)\,a^3xy\,(x-\frac{zx^4}{a^3y}+\frac{z^2x^7}{a^3y^2}\ \&c.$$

$$\begin{array}{l}
a^3xy+zx^4 \\
\hline
\quad -zx^4+0 \\
\quad -zx^4-\frac{z^2x^7}{a^3y} \\
\hline
\qquad +\frac{z^2x^7}{a^3y}+\&c. \\
\hline
\end{array}$$

Therefore $\frac{a^3xy}{a^3y+zx^3}=x-\frac{zx^4}{a^3y}+\frac{z^2x^7}{a^6y^2}$ &c.

EXAM. 18.

Find the value of $\frac{x-3y}{x-4y}r$ *in an infinite ſeries.*

$$x-4y)\,x-3y\,(1+\frac{y}{x}+\frac{4y^2}{x^2}+\frac{16y^3}{x^3}+\frac{64y^4}{x^4}\ \&c.$$

$$\begin{array}{l}
\underline{x-4y} \\
+y+0 \\
\underline{+y-\frac{4y^2}{x}} \\
\qquad +\frac{4y^2}{x}+0 \\
\qquad \underline{+\frac{4y^2}{x}-\frac{16y^3}{x^2}} \\
\qquad\qquad +\frac{16y^3}{x^2}+0 \\
\qquad\qquad \underline{+\frac{16y^3}{x^2}-\frac{64y^4}{x^3}} \\
\qquad\qquad\qquad \underline{+\frac{64y^4}{x^3}\ \&c.}
\end{array}$$

Therefore $\frac{x-3y}{x-4y}r = r\times:1+\frac{y}{x}+\frac{4y^2}{x^2}+\frac{16y^3}{x^3}+\frac{64y^4}{x^4}$ &c.

Corollary.

Hence by the ſame method of proceeding.

1. $\frac{a^2r}{r^2\mp b\times x}=\frac{aa}{r}\times:1\pm\frac{bx}{r}+\frac{b^2x^2}{r^4}\pm\frac{b^3x^3}{r^6}+\frac{b^4x^4}{r^8}$ &c.

2. And $\frac{q^2r^2}{r^3 \mp a \times \overline{nz+my}} = \frac{q^2}{r} \times 1 \pm \frac{a}{r} \times \frac{nz+my}{r^2} + \frac{a^2}{r^2} \times \frac{\overline{nz+my}|^2}{r^4} \pm \frac{a^3}{r^3} \times \frac{\overline{nz+my}|^3}{r^6}$ &c.

EXAM. 19.

Reduce $\frac{\frac{1}{2}bcc}{acc+bx^2}$ *into a ſeries.*

$$acc+bx^2)\tfrac{1}{2}bcc\ldots\ldots\left(\frac{b}{2a} - \frac{b^2x^2}{2a^2c^2} + \frac{b^3x^4}{2a^3c^4} - \frac{b^4x^6}{2a^4c^6} + \frac{b^5x^8}{2a^5c^8}\right. \text{ \&c.}$$

$$\underline{\tfrac{1}{2}bcc + \frac{b^2x^2}{2a}}$$

$$-\frac{b^2x^2}{2a} + 0$$

$$\underline{-\frac{b^2x^2}{2a} - \frac{b^3x^4}{2a^2c^2}}$$

$$\underline{+\frac{b^3x^4}{2a^2c^2} + \frac{b^4x^6}{2a^3c^4}}$$

$$-\frac{b^4x^6}{2a^3c^4} + 0$$

$$\underline{-\frac{b^4x^6}{2a^3c^4} - \frac{b^5x^8}{2a^4c^6}}$$

$$\underline{+\frac{b^5x^8}{2a^4c^6} \text{ \&c.}}$$

Therefore $\frac{\frac{1}{2}bcc}{acc+bxx} = \frac{b}{2a} - \frac{b^2x^2}{2a^2c^2} + \frac{b^3x^4}{2a^3c^4} - \frac{b^4x^6}{2a^4c^6} + \frac{b^5x^8}{2a^5c^8}$ &c. $= \frac{b}{2a} \times 1 - \frac{bx^2}{ac^2} + \frac{b^2x^4}{a^2c^4} - \frac{b^3x^6}{a^5c^6} + \frac{b^4x^0}{a^4c^8}$ &c.

Proof.

Proof.

$$\text{Multiply } \frac{b}{2a} - \frac{b^2x^2}{2a^2c^2} + \frac{b^3x^4}{2a^3c^4} - \frac{b^4x^6}{2a^4c^6} + \frac{b^5x^8}{2a^5c^8} \text{ \&c.}$$

$$\text{by } \underline{acc + bx^2}$$

$$\tfrac{1}{2}bc^2 - \frac{b^2x^2}{2a} + \frac{b^3x^4}{2a^2c^2} - \frac{b^4x^6}{2a^3c^4} + \frac{b^5x^8}{2a^4c^6} \text{ \&c.}$$

$$+ \frac{b^2x^2}{2a} - \frac{b^3x^4}{2a^2c^2} + \frac{b^4x^6}{2a^3c^4} - \frac{b^5x^8}{2a^4c^6} \text{ \&c.}$$

$$\tfrac{1}{2}bcc \;.\; 0 \;.\; 0 \;.\; 0 \;.\; 0 \text{ dividend}$$

Mr. *Emerſon*, at page 26 of his Trigonometry, from a, a given arch, finds its ſine $x = a - \frac{a^3}{6} + \frac{1}{120}a^5 - \frac{a^7}{5040}$ &c.

And its coſine $y = 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{1}{720}a^6$ &c.

Now $\frac{\text{ſine}}{\text{coſine}} =$ tangent; and $\frac{\text{radius ſquare}}{\text{coſine}} =$ ſecant,

And $\frac{\text{radius ſquare}}{\text{tangent}} =$ cotangent. See my book of Fortification and Gunnery, page 136.

EXAM. 20.

Put $\dfrac{1 = \textit{radius square.}}{1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} \textit{ \&c. cosine}}$ *into a series.*

$1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6)\,1\ \ (1 + \frac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6$ &c. secant.

$$\begin{array}{r}
1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6 \\
\hline
+ \frac{1}{2}a^2 - \frac{1}{24}a^4 + \frac{1}{720}a^6 \\
+ \frac{1}{2}a^2 - \frac{1}{4}a^4 + \frac{1}{48}a^6 \\
\hline
+ \frac{5}{24}a^4 - \frac{14}{720}a^6 \text{ \&c.} \\
+ \frac{5}{24}a^4 - \frac{5}{48}a^6 \text{ \&c.} \\
\hline
+ \frac{61}{720}x^6 \text{ \&c.} \\
\text{\&c.} \\
\hline
\end{array}$$

Explanation.

1. $\frac{1}{4}a^4 - \frac{1}{24}a^4 = \frac{24a^4 - 4a^4}{96} = \frac{20a^4}{96} = \frac{5}{24}a^4$

2. $\frac{1}{720}a^6 - \frac{1}{48}a^6 = -\frac{720a^6 - 48a^6}{34560} = -\frac{672a^6}{34560} = -\frac{14}{720}a^6$.

3. $\frac{5}{48}a^6 - \frac{14}{720}a^6 = \frac{3600a^6 - 672a^6}{34560} = \frac{2928a^6}{34560} = \frac{61}{720}a^6$.

Therefore $\dfrac{1}{1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6 \text{ \&c.}} = 1 + \frac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6$ &c. = secant.

Proof.

Proof.

$$\begin{array}{rl}
\text{Multiply} & 1+\frac{1}{2}a^2+\frac{5}{24}a^4+\frac{61}{720}a^6\ \&c.\\
\text{by} & 1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6\ \&c.\\
\hline
 & 1+\frac{1}{2}a^2+\frac{5}{24}a^4+\frac{61}{720}a^6\ \&c.\\
 & \quad -\frac{1}{2}a^2-\frac{1}{4}a^4-\frac{5}{48}a^6\ \&c.\\
 & \qquad\qquad +\frac{1}{24}a^4+\frac{1}{48}a^6\ \&c.\\
 & \qquad\qquad\qquad -\frac{1}{720}a^6\ \&c.\\
\hline
 & 0\ .\ 0\ .\ 0\ .\ 0\quad \text{dividend.}\\
\hline
\end{array}$$

EXAM. 21.

Find $\dfrac{a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7\ \&c.\ \textit{ſine}}{1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6\ \&c.\ \textit{coſine}}$ *in a ſeries.*

$1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6$ &c.$)a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7$ &c.$(a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7$ &c. tangent.

$$\begin{array}{r}
a-\frac{1}{2}a^3+\frac{1}{24}a^5-\frac{1}{720}a^7\ \&c.\\
\hline
+\frac{1}{3}a-\frac{1}{30}a^5+\frac{1}{840}a^7\ \&c.\\
+\frac{1}{3}a^3-\frac{1}{6}a^5+\frac{1}{720}a^7\ \&c.\\
\hline
\frac{2}{15}a^5-\frac{4}{315}a^7\ \&c.\\
\frac{2}{15}a^5-\frac{2}{30}a^7\ \&c.\\
\hline
+\frac{17}{315}a^7\ \&c.\\
\hline
\end{array}$$

Operation explained.

1. $\frac{1}{6}a^3-\frac{1}{2}a^3=\frac{1}{3}a^3$.

2. $\frac{1}{120}a^5-\frac{1}{24}a^4=\frac{120x^5-24a^5}{2880}=\frac{96a^5}{2880}=$
$-\frac{1}{30}a^5$.

3. $\frac{1}{5040}a^7-\frac{1}{720}a^7=\frac{5040a^7-720a^7}{3628800}=$
$\frac{4320}{3628800}a^7=\frac{1}{840}a^7$.

$$4.\ \frac{-4}{315}+\frac{2}{30}=\frac{630a^7-120a^7}{9450}=\frac{510a^7}{9450}=\frac{17a^7}{315}.$$

Proof.

$$\text{Mult. } a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7\ \&c.$$

$$\text{by } 1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^7\ \&c.$$

$$\begin{array}{rrrr}
a+\frac{1}{3}a^3 & +\frac{2}{15}a^5 & +\frac{17}{315}a^7 & \&c. \\
-\frac{1}{2}a^3 & -\frac{1}{6}a^5 & -\frac{1}{15}a^7 & \&c. \\
 & +\frac{1}{24}a^5 & +\frac{1}{72}a^7 & \&c. \\
 & & -\frac{1}{720}a^7 & \&c.
\end{array}$$

$$a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7\ \&c.\ \text{dividend.}$$

The coefficients of x^7 are collected thus:

$$1.\ \frac{17}{315}+\frac{1}{72}=\frac{1539}{22680}=\frac{19}{280}.$$

$$2.\ -\frac{1}{15}-\frac{1}{720}=-\frac{735}{10800}=-\frac{49}{720}.$$

$$3.\ \text{Then } \frac{19}{280}-\frac{49}{720}=\frac{-13720+13680}{201600}=\frac{-40}{201600}=-\frac{1}{5040}.$$

Therefore

$$\text{Therefore } \frac{a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7 \text{ \&c. ſine}}{1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^7 \text{ \&c. coſine}}$$

$$\text{is} = a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 \text{ \&c. the tangent.}$$

EXAM. 22.

Find the value of $\dfrac{1 = \textit{rad. ſquare.}}{a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 \text{ \&c. } \textit{tang.}}$ *in a ſeries.*

$$a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 \text{\&c.}\Big)\,1 \ldots.\Big(\frac{1}{a} - \frac{a}{3} - \frac{a^3}{45}$$

$$- \frac{2}{945}a^5 - \frac{a^7}{4725} \text{ \&c. cotangent.}$$

$$\begin{array}{r}
1 + \frac{a^2}{3} + \frac{2a^4}{15} + \frac{17}{315}a^6 \\
\hline
- \frac{a^2}{3} - \frac{2a^4}{15} - \frac{17a^6}{315} \\
- \frac{a^2}{3} - \frac{a^4}{9} - \frac{2a^6}{45} \text{ \&c.} \\
\hline
- \frac{a^4}{45} - \frac{31a^6}{315} \text{ \&c.} \\
- \frac{a^4}{45} - \frac{a^6}{135} \text{ \&c.} \\
\hline
- \frac{2a^6}{945} \text{ \&c.} \\
- \frac{2a^6}{945} \text{ \&c.}
\end{array}$$

Proof.

Proof.

$$\begin{array}{rl}
\text{Mult.} & \frac{1}{a} - \frac{1}{3}a - \frac{1}{45}a^3 - \frac{2}{945}a^5 - \frac{1}{4725}a^7 \text{ \&c.} \\
\text{by} & a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 \text{ \&c.} \\
\hline
 & 1 - \frac{1}{3}a^2 - \frac{1}{45}a^4 - \frac{2}{945}a^6 - \frac{1}{4725}a^8 \text{ \&c.} \\
 & + \frac{1}{3}a^2 - \frac{1}{9}a^4 - \frac{1}{135}a^6 - \frac{2}{2835}a^8 \text{ \&c.} \\
 & + \frac{2}{15}a^4 - \frac{2}{45}a^6 - \frac{2}{675}a^8 \text{ \&c.} \\
 & + \frac{17}{315}a^6 - \frac{17}{945}a^8 \text{ \&c.} \\
\hline
 & 1 \;.\; 0 \;.\; 0 \;.\; 0 \;.\; 0 \text{ divid.} \\
\hline
\end{array}$$

Therefore $\dfrac{1}{a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 \text{ \&c.}} =$

$\dfrac{1}{a} - \dfrac{a}{3} - \dfrac{a^3}{45} - \dfrac{2a^5}{945} - \dfrac{a^7}{4725}$ &c. the cotangent.

Corollary.

Hence if a be a given arch; x the ſine, y the coſine, t tangent, &c. then

Its ſine is $x = a - \dfrac{a^3}{6} + \dfrac{1}{120}a^5 - \dfrac{a^7}{5040}$ &c.

Coſine $y = 1 - \dfrac{a^2}{2} + \dfrac{a^4}{24} - \dfrac{1}{720}a^6$ &c.

Tangent $t = a + \dfrac{1}{3}a^3 + \dfrac{2}{15}a^5 + \dfrac{17}{315}a^7$ &c.

Cotangent $\tau = \frac{1}{a} - \frac{a}{3} - \frac{a^3}{45} - \frac{2a^5}{945} - \frac{a^7}{4725}$ &c.

Secant $s = 1 + \frac{1}{2}a^2 + \frac{5}{24}a^4 + \frac{61}{720}a^6$ &c.

Quantities may be raiſed to any powers, and reduced into an infinite ſeries without the trouble of a long diviſion, by certain Theorems for that purpoſe, ſome of which I ſhall ſet down as under; they are general, and require no more than the bare ſubſtituting of quantities.

THEOREM I.

$$\overline{A + 2B}|^m = A^m + mA^{m-1} \times 2B + \frac{m}{1} . \frac{m-1}{2} A^{m-2} \times 4B^2 + \frac{m.m-1.m-2}{1.2.3} A^{m-3} \times 8B^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4} A^{m-4} \times 16B^4 + \&c.$$

THEOREM II.

$$\overline{A + 3B}|^m = A^m + mA^{m-1} \times 3B + \frac{m.m-1}{1.2} A^{m-2} \times 9B^2 + \frac{m.m-1.m-2}{1.2.3} A^{m-3} \times 27B^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4} A^{m-4} \times 81B^4 + \&c.$$

THEOREM III.

$$\overline{A + Bx^n}|^m = A^m + mBA^{m-1}x^n + \frac{m.m-1}{1.2}$$

$$B^2A^{m-2}x^{2n} + \frac{m.\overline{m-1}.\overline{m-2}}{1.2.3}B^3A^{m-3}x^{3n} + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{1.2.3.4}B^4A^{m-4}x^{4n} + \&c.$$

THEOREM IV.

$$\frac{1}{\overline{a+b}|^n} = \frac{1}{a^n} - \frac{nb}{a^{n+1}} + \frac{n.\overline{n+1}.b^2}{1.2.a^{n+2}} - \frac{n.\overline{n+1}.\overline{n+2}.b^3}{1.2.3.a^{n+3}} + \frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.b^4}{1.2.3.a^{n+4}} - \&c.$$

Or thus.

$$\frac{1}{\overline{a+b}|^n} \text{ or } \overline{a+b}|^{-n} = \overset{A}{\frac{1}{a^n}} - \frac{n}{1}\times\overset{B}{\frac{b}{a}}A + \frac{n+1}{2}\times\overset{C}{\frac{b}{a}}B - \frac{n+2}{3}\times\overset{D}{\frac{b}{a}}C + \frac{n+3}{4}\times\overset{E}{\frac{b}{a}}D - \&c.$$

THEOREM V.

$$\frac{1}{\overline{a-b}|^n} = \frac{1}{a^n} + \frac{n}{1}\times\frac{b}{a}A + \frac{n+1}{2}\times\frac{b}{a}B + \frac{n+2}{3}\times\frac{b}{a}C + \frac{n+3}{4}\times\frac{b}{a}D + \&c.$$

where A, B, C, &c. denote the foregoing terms with their signs, and n is any index.

THEOREM VI.

$$\frac{aa}{\overline{A-B}|^n} \text{ or } aa\times\overline{A-B}|^{-n} = aaA^{-n} + \frac{n}{1}aaA^{-n-1}B + \frac{n.\overline{n+1}}{1.2}aaA^{-n-2}B^2 + \frac{n.\overline{n+1}.\overline{n+2}}{1.2.3}$$

aa

$$aaA^{-n-3}B^3 + \&c. = \frac{aa}{A^n} + \frac{naaB}{A^{n+1}} + \frac{\overline{nn+n}.aaB^2}{2A^{n+2}}$$

$$+ \frac{\overline{n^3 + 3n^2 + 2n} \times aaB^3}{6A^{n+3}} + \&c.$$

THEOREM VII.

$$\frac{\overline{A+B}^{|n}}{cc} = \frac{A^n}{cc} + \frac{nA^{n-1}B}{cc} + \frac{n.n-1.A^{n-2}B^2}{1\,.\,2\,.\,cc}$$

$$+ \frac{n.n-1.n-2A^{n-3}B^3}{1\,.\,2\,.\,3\,.\,cc} + \frac{n.n-1.n-2.n-3A^{n-4}B^4}{1\,.\,2\,.\,3\,.\,4\,.\,cc}$$

$$+ \&c.$$

THEOREM VIII.

$$\frac{a\,.\,\overline{c-x}^{|m} + b\,.\,\overline{c-x}^{|n}}{d^3} =$$

$$\frac{ac^m + bc^n - maxc^{m-1} - nbxc^{n-1}}{d^3} +$$

$$\underline{\frac{m.m-1}{1\,.\,2}\,ax^2c^{m-2} + \frac{n.n-1}{1\,.\,2}\,bx^2c^{n-2}\ \&c.}$$

THEOREM IX.

$$\overline{a-x}^{|m-1} = a^{m-1} - \overline{m-1} \times xa^{m-2} + \overline{m-1}$$

$$\times \frac{m-2}{2}\,x^2a^{m-3} - \frac{m-1}{1} \times \frac{m-2}{2} \times \frac{m-3}{3} \times$$

$$x^3a^{m-4}\ \&c.$$

Theſe Theorems are general, and very uſeful to ſhew the generation of powers, both with regard to raiſing of quantities to any known power, m or n, or to reduce them into an infinite ſeries; the uſe of which I will give an inſtance or two by the following examples.

EXAMPLE I.

Let $\frac{1}{\overline{a - x}|^{2}}$ *be put into an infinite ſeries.*

This belongs to Theorem VI.

Here $aa = 1$; $A = a$, $B = x$; $B^2 = x^2$, $B^3 = x^3$, &c. and $n = 2$.

Then 1. $\frac{aa}{A^n} = \frac{1}{a^2}$,

2. $\frac{naaB}{A^{n+1}} = \frac{2.1x}{a^3}$,

3. $\frac{\overline{nn+n} \times aaB^2}{2A^{n+2}} = \frac{6.1x^2}{2a^4} = \frac{3.1x^2}{a^4}$,

4. $\frac{\overline{n^3+3n^2+2n} \times aaB^3}{6A^{n+3}} = \frac{24.1x^3}{6a^5} = \frac{4.1x^3}{a^5}$,

&c. &c.

Therefore $\frac{1}{\overline{a - x}|^{2}} = \frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} + \frac{4x^3}{a^5}$ &c.

EXAM. 2.

Put this expreſſion $\frac{1}{\overline{a - 2x}|^{2}}$ *into a ſeries.*

Here $aa = 1$, $A = a$, $B = 2x$, $B^2 = 4x^2$, $B^3 = 8x^3$, &c. and $n = 2$. *Theo.* 6.

Then 1. $\frac{aa}{A^n} = \frac{1}{aa}$.

2. $\frac{naaB}{A^{n+1}} = \frac{2.2x}{a^3}$.

3. $\frac{\overline{nn+n} \times aaB^2}{2A^{n+2}} = \frac{6.4x^2}{2a^4} = \frac{3.4x^2}{a^4}$,

4. $\frac{\overline{n^3+3n^2+2n} \times aaB^3}{6A^{n+3}} = \frac{24.8x^3}{6a^5} = \frac{4.8x^3}{a^5}$ &c.

Therefore $\frac{1}{\overline{a-2x}|^2} = \frac{1}{aa} + \frac{2.2x}{a^3} + \frac{3.4x^2}{a^4} + \frac{4.8x^3}{a^5}$ &c.

Exam. 3.

Find the value of $\frac{1}{\overline{a-3x}|^2}$ *in an infinite ſeries.*

By *Theorem* 6.

Here $aa = 1$, $A = a$, $B = 3x$, then $B^2 = 9x^2$, $B^3 = 27x^3$, and $n = 2$.

Then 1. $\frac{aa}{A^n} = \frac{1}{aa}$.

2. $\frac{naaB}{A^{n+1}} = \frac{2.3x}{a^3}$.

3. $\frac{\overline{nn+n} \times aaB^2}{2A^{n+2}} = \frac{6.9x^2}{2a^4} = \frac{3.9x^2}{a^4}$.

4. $\frac{\overline{n^3+3n^2+2n} \times aaB^3}{6A^{n+3}} = \frac{24.27x^3}{6a^5} = \frac{4.27x^3}{a^5}$.

&c.

Therefore $\frac{1}{\overline{a-3x}|^2} = \frac{1}{aa} + \frac{2.3x}{a^3} + \frac{3.9x^2}{a^4} + \frac{4.27x^3}{a^5}$ + &c.

Exam. 4.

Find the value of $\frac{1}{\overline{a-3x}|^3}$ *in an infinite ſeries.*

Here $aa = 1$, $A = a$, $B = 3x$, then $B^2 = 9x^2$, $B^3 = 27x^3$, and $n = 3$.

1. aa

1. $\frac{aa}{A^n} = \frac{1}{a^3}$.

2. $\frac{na^2B}{A^{n+1}} = \frac{3.3x}{a^4}$.

3. $\frac{\overline{nn+n}\times aaB^2}{2A^{n+2}} = \frac{12.9x^2}{2.a^5} = \frac{6.9x^2}{a^5}$.

4. $\frac{n^3+Bn^2+2n\times aaB^3}{6A^{n+3}} = \frac{60.27x^3}{6a^6} = \frac{10.27x^3}{a^6}$,

&c.

Hence $\frac{1}{\overline{a-3x}|^3} = \frac{1}{a^3} + \frac{3.3x}{a^4} + \frac{6.9x^2}{a^5} + \frac{10.27x^3}{a^6} + \&c.$

I now proceed to ſhew how a ſurd may be reduced into an infinite ſeries, by extracting its root as in common Arithmetic.

EXAM. I.

Find the value of $\sqrt{1-\frac{1}{2}x}$, *in an infinite ſeries; or to extract the ſquare root of* $1-\frac{1}{2}x$.

$$1-\tfrac{1}{2}x\left(1-\frac{x}{4}-\frac{x^2}{32}-\frac{x^3}{128}-\&c.\right.$$

$$\underline{1}$$

$$\left.2-\frac{x}{4}\right)-\tfrac{1}{2}x$$

$$\times-\frac{x}{4}=\underline{-\tfrac{1}{2}x+\frac{xx}{16}}$$

$$2-\frac{x}{2}-\frac{x^2}{32}\left(-\frac{xx}{16}\right.$$

$$\times-\frac{x^2}{32}=\underline{-\frac{xx}{16}+\frac{x^3}{64}\ \&c.}$$

2 —

$$2-\frac{x}{2}-\frac{x^2}{16}-\frac{3x^3}{384}\Big)-\frac{x^3}{64}\ \&c.$$

$$\times-\frac{3x^3}{384}=-\frac{x^3}{64}\ \&c.$$

Now if the root $1-\frac{x}{4}-\frac{x^2}{32}-\frac{x^3}{128}-$ &c. be ſquared, it will produce $1-\frac{1}{2}x$, which proves the work to be right.

$$\begin{array}{rrrrl}
\text{Thus }1-\frac{x}{4} & -\frac{x^2}{32} & -\frac{x^3}{128} & -\ \&c. \\
1-\frac{x}{4} & -\frac{x^2}{32} & -\frac{x^3}{128} & -\ \&c. \\
\hline
1-\frac{x}{4} & -\frac{x^2}{32} & -\frac{x^3}{128} & -\ \&c. \\
-\frac{x}{4} & +\frac{x^2}{16} & +\frac{x^3}{128} & \&c. \\
 & -\frac{x^2}{32} & +\frac{x^3}{128} & \&c. \\
 & & -\frac{x^3}{128} & \&c. \\
\hline
1-\frac{1}{2}x & .\ 0 & .\ 0 & \\
\hline
\end{array}$$

I neglect all ſuch terms whoſe dimenſions exceed thoſe of the laſt term $\frac{3x^3}{384}$, to which the root is to be continued.

Therefore $\sqrt{1-\frac{1}{2}x}=1-\frac{x}{4}-\frac{x^2}{32}-\frac{x^3}{128}$ — &c.

EXAM.

EXAM. 2.

Find the value of $\sqrt{aa + xx}$, *in an infinite series; or to extract the square root of* $aa + xx$.

$$aa + xx\left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\right. + \text{\&c.}$$

$$aa$$

$$\left.2a + \frac{xx}{2a}\right) + xx$$

$$\times \frac{xx}{2a} = xx + \frac{x^4}{4a^2}$$

$$2a + \frac{xx}{a} - \frac{x^4}{8a^3}\left(- \frac{x^4}{4a^2}\right.$$

$$\times - \frac{x^4}{8a^3} = - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$

$$\left.2a + \frac{xx}{a}\right) + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$\times + \frac{x^6}{16a^5} = \frac{x^6}{8a^4} + \frac{x^8}{16a^6} \text{ \&c.}$$

$$- \frac{5x^8}{64a^6}$$

Therefore $\sqrt{aa+xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} +$ &c.

Mult. $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

by $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

$aa +$

$$\begin{array}{llllll}
aa + \frac{x^2}{2} & - \frac{x^4}{8a^2} & + \frac{x^6}{16a^4} & - \frac{5x^8}{128a^6} & \&c. \\
\quad + \frac{x^2}{2} & + \frac{x^4}{4a^2} & - \frac{x^6}{16a^4} & + \frac{x^8}{32a^6} & \&c. \\
 & - \frac{x^4}{8a^2} & - \frac{x^6}{16a^4} & + \frac{x^8}{64a^6} & \&c. \\
 & & + \frac{x^6}{16a^4} & + \frac{x^8}{32a^6} & \&c. \\
 & & & - \frac{5x^8}{64a^6} & \&c. \\
\hline
aa + xx & . \; 0 & . \; 0 & . \; 0 & \\
\hline
\end{array}$$

Corollary.

Hence $\sqrt{aa - xx} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

And $\sqrt{rr - xx} = r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}$ &c.

Alſo $\sqrt{xx + aa} = x + \frac{a^2}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5} - \frac{5a^6}{128x^7}$ &c.

EXAM.

EXAM. 3.

Let it be required to extract the ſquare root of $\frac{1}{1-xx}$, *or to expreſs the value of* $\sqrt{\frac{1}{1-xx}}$, *in an infinite ſeries.*

Before we can extract the root of $\frac{1}{1-xx}$, it muſt be put into a ſeries by diviſion. See *Ex.* 9. *Cor.*

Now $1-xx)1(1+x^2+x^4+x^6+x^8$ &c. by diviſion.

Then $1+x^2+x^4+x^6+x^8$ &c.$\left(1+\frac{x^2}{2}+\frac{3x^4}{8}+\frac{5x^6}{16}+\frac{35x^8}{128}\right.$ &c. root.

$$\begin{array}{l}
1 \\
\hline
2+\frac{xx}{2}\Big)+x^2+x^4 \\
\qquad\times\frac{xx}{2}=+x^2+\frac{x^4}{4} \\
\hline
2+x^2+\frac{3x^4}{8}\Big)+\frac{3x^4}{4}+x^6+x^8+x^{10} \\
\qquad\qquad\times\frac{3x^4}{8}=+\frac{3x^4}{4}+\frac{3x^6}{8}+\frac{9x^8}{64} \\
\hline
\qquad\qquad\qquad 2+x^2\Big)\frac{5x^6}{8}+\frac{55x^8}{64} \\
\qquad\qquad\qquad\times\frac{5x^6}{16}=\frac{5x^6}{8}+\frac{5x^8}{16} \\
\hline
\qquad\qquad\qquad\qquad\qquad+\frac{35x^8}{64}\ \text{\&c.} \\
\hline
\end{array}$$

Operation explained.

1. $2+\frac{x^2}{2}\times\frac{x^2}{2}=\frac{2x^2}{2}+\frac{x^4}{4}=x^2+\frac{x^4}{4}$.

2. 2+

2. $2 + x^2 + \frac{3x^4}{8} \times \frac{3x^4}{8} = \frac{6x^4}{8} + \frac{3x^6}{8} + \frac{9x^8}{64}$ $= \frac{3x^4}{4} + \frac{3x^6}{8}$ &c.

3. $2 + x^2 \times \frac{5x^6}{16} = \frac{10x^6}{16} + \frac{5x^8}{16}$ &c. $= \frac{5x^6}{8} + \frac{5x^8}{16}$ &c.

And $\frac{55x^8}{64} - \frac{5x^8}{16} = \frac{560x^8}{1024} = \frac{35x^8}{64}$.

Therefore $\sqrt{\frac{1}{1 - xx}} = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \frac{35x^8}{128}$ &c.

EXAM. 4.

Put $\sqrt{aa - 2bx - xx}$, *into an infinite ſeries.*

$$aa - 2bx - xx\left(a - \frac{bx}{a} - \frac{x^2}{2a} - \frac{b^2x^2}{2a^3} - \frac{bx^3}{2a^3} - \frac{b^3x^3}{2a^5} \text{ \&c.}\right.$$

$$\underline{aa}$$

$$2a - \frac{bx}{a}\Big) - 2bx - xx$$

$$\times - \frac{bx}{a} = -2bx + \frac{b^2x^2}{a^2}$$

$$2a - \frac{2bx}{a}\Big) - xx - \frac{b^2x^2}{a^2}$$

$$\times - \frac{x^2}{2a} = -xx + \frac{bx^2}{a^2}$$

&c. &c.

Therefore $\sqrt{aa - 2bx - xx} = a - \frac{bx}{a} - \frac{x^2}{2a}$

$\frac{b^2x^2}{2a^3} - \frac{bx^3}{2a^3} - \frac{b^3x^3}{2a^5}$ &c. $= a - \frac{bx}{a} - \overline{\frac{1}{2a} - \frac{b^2}{2a^3}}$

$\times x^2 - \overline{\frac{b}{2a^3} - \frac{b^3}{2a^5}} \times x^3$ &c.

EXAM. 5.

Find the value of $\sqrt{a^2 - b^2x^2}$, *in an infinite ſeries.*

$$a^2 - b^2x^2 \left(a - \frac{b^2x^2}{2a} - \frac{b^4x^4}{8a^3} - \frac{b^6x^6}{16a^5} - \frac{15b^8x^8}{256a^7} \right. \text{ \&c.}$$

$$\underline{a^2 \qquad\qquad\qquad}$$

$$\left. 2a - \frac{b^2x^2}{2a} \right) - b^2x^2$$

$$\times - \frac{2^2x^2}{2a} = - b^2x^2 + \frac{b^4x^4}{4a^2}$$

$$\left. 2a - \frac{b^2x^2}{a} - \frac{b^4x^4}{8a^3} \right) \overline{- \frac{b^4x^4}{4a^2}}$$

$$\times - \frac{b^4x^4}{8a^3} = - \frac{b^4x^4}{4a^2} + \frac{b^6x^6}{8a^4} + \frac{b^8x^8}{64a^6}$$

$$\left. 2a - \frac{b^2x^2}{a} \right) \overline{- \frac{b^6x^6}{8a^4} - \frac{b^8x^8}{64a^6}}$$

$$\times - \frac{b^6x^6}{16a^5} = - \frac{b^6x^6}{8a^4} + \frac{b^8x^8}{16a^6} \text{ \&c.}$$

$$\overline{\qquad\qquad - \text{ \&c.}}$$

Therefore $\sqrt{a^2 - b^2x^2} = a - \frac{b^2x^2}{2a} - \frac{b^4x^4}{2.4a^3} -$

$\frac{b^6x^6}{16a^5} - \frac{3.5b^8x^8}{4.8.8a^7}$ &c.

Multiply

$$\begin{array}{lcccccc}
\text{Mult.} & a - \frac{b^2x^2}{2a} & - \frac{b^4x^4}{8a^3} & - \frac{b^6x^6}{16a^5} & - \frac{15b^8x^8}{256a^7} & \text{\&c.} \\
\text{by} & a - \frac{b^2x^2}{2a} & - \frac{b^4x^4}{8a^3} & - \frac{b^6x^6}{16a^5} & - \frac{15b^8x^8}{256a^7} & \text{\&c.} \\
\hline
 & aa - \frac{b^2x^2}{2} & - \frac{b^4x^4}{8a^2} & - \frac{b^6x^6}{16a^4} & - \frac{15b^8x^8}{256a^6} & \text{\&c.} \\
 & - \frac{b^2x^2}{2} & + \frac{b^4x^4}{4a^2} & + \frac{b^6x^6}{16a^4} & + \frac{b^8x^8}{32a^6} & \text{\&c.} \\
 & & - \frac{b^4x^4}{8a^2} & + \frac{b^6x^6}{16a^4} & + \frac{b^8x^8}{64a^6} & \text{\&c.} \\
 & & & - \frac{b^6x^6}{16a^4} & + \frac{b^8x^8}{32a^6} & \text{\&c.} \\
 & & & & - \frac{15b^8x^8}{256a^6} & \text{\&c.} \\
\hline
\text{Product} & a^2 - b^2x^2 & . \; 0 & . \; 0 & . \; 0 & \\
\hline
\end{array}$$

EXAM. 6.

Find the value of $\frac{x}{a}\sqrt{aa + 4xx}$, *in an infinite series.*

1. $\sqrt{aa+4xx} = a + \frac{2x^2}{a} - \frac{2x^4}{a^3} + \frac{4x^6}{a^5}$ &c. by *Ex.* 2.

2. Then $\frac{x}{a} \times : a + \frac{2x^2}{a} - \frac{2x^4}{a^3} + \frac{4x^6}{a^5}$ &c. $= x + \frac{2x^3}{a^2} - \frac{2x^5}{a^4} + \frac{4x^7}{a^6}$ &c.

Therefore $\frac{x}{a}\sqrt{aa + 4xx} = x + \frac{2x^3}{a^2} - \frac{2x^5}{a^4} + \frac{4x^7}{a^6}$ &c.

EXAM. 7.

Put $\frac{aa}{b+x} + \sqrt{cc+xx}$, *into an infinite ſeries.*

1. $\frac{aa}{b+x} = \frac{aa}{b} - \frac{a^2x}{bb} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5}$ &c. by diviſion.

2. $\sqrt{cc+xx} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5}$ &c. by *Ex.* 2.

Add the two ſeries together, and the ſum is = $\frac{aa}{b} + c - \frac{aax}{b^2} + \frac{a^2x^2}{b^3} + \frac{x^2}{2c} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5} - \frac{x^4}{8c^3}$ &c.

Therefore $\frac{aa}{b+x} + \sqrt{cc+xx} = \frac{aa}{b} + c - \frac{a^2x}{b^2} + \overline{\frac{a^2}{b^3} + \frac{1}{2c}} \times x^2 - \frac{a^2x^3}{b^4} + \overline{\frac{a^2}{b^5} - \frac{1}{8c^3}} \times x^4$ &c.

EXAM. 8.

Extract the ſquare root of $rr - xx + \frac{x^4}{3r^2} - \frac{2x^6}{45r^4}$ &c.

$$\begin{array}{l}
rr - xx + \frac{x^4}{3r^2} - \frac{2x^6}{45r^4}\ \&c. \Big(r - \frac{x^2}{2r} + \frac{x^4}{24r^3} - \frac{x^6}{720r^5}\ \&c. \\
rr \\
\hline
2r - \frac{xx}{2r} \Big)\ - xx + \frac{x^4}{3r^2} \\
\times - \frac{xx}{2r} = - xx + \frac{x^4}{4r^2} \\
\hline
\end{array}$$

2r —

$$2r - \frac{xx}{r}\Big(+ \frac{x^4}{12r^2} - \frac{2x^6}{45r^4} \text{ \&c.}$$

$$\times + \frac{x^4}{24r^3} = + \frac{x^4}{12r^2} - \frac{x^6}{24r^4} \text{ \&c.}$$

$$2r - \frac{xx}{r}\Big) - \frac{x^6}{360r^4} \text{ \&c.}$$

$$\times - \frac{x^6}{720r^5} = - \frac{x^6}{360r^4} \text{ \&c.}$$

&c.

Therefore $\sqrt{rr - xx + \frac{x^4}{3r^2} - \frac{2x^6}{45r^4} \text{ \&c.}} = r - \frac{xx}{1.2r} + \frac{x^4}{1.2.3.4r^3} - \frac{x^6}{1.2.3.4.5.6r^5}$ &c.

EXAM. 9.

Find the value of $\sqrt[4]{1 - xx}$, *or the biquadrate root of* $1 - xx$.

$$1 - xx\Big(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} \text{ \&c.}$$ ſquare root.

$$1$$

$$2 - \frac{xx}{2}\Big) - xx$$

$$\times - \frac{xx}{2} = - xx + \frac{x^4}{4}$$

$$2 - x^2 - \frac{x^4}{8}\Big(- \frac{x^4}{4}$$

$$\times - \frac{x^4}{8} = - \frac{x^4}{4} + \frac{x^6}{8} + \frac{x^8}{64}$$

$$2 - x^2\,\Big)\ -\frac{x^6}{8} - \frac{x^8}{64}$$

$$\times -\frac{x^6}{16} = -\frac{x^6}{8} + \frac{x^8}{16}\ \text{\&c.}$$

$$-\frac{5x^8}{64}\ \text{\&c.}$$

Again.

$$1 - \frac{xx}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}\ \text{\&c.}\ \Big(1 - \frac{x^2}{4} - \frac{3x^4}{32} - \frac{7x^6}{128} - \frac{77x^8}{2048}\ \text{\&c.}$$

$$1$$

$$2 - \frac{xx}{4}\Big)\ -\frac{xx}{2} - \frac{x^4}{8}$$

$$\times -\frac{xx}{4} = -\frac{xx}{2} + \frac{x^4}{16}$$

$$2 - \frac{xx}{2} - \frac{3x^4}{32}\Big(\ -\frac{3x^4}{16} - \frac{x^6}{16} - \frac{5x^8}{128}\ \text{\&c.}$$

$$\times -\frac{3x^4}{32} = -\frac{3x^4}{16} + \frac{3x^6}{64} + \frac{9x^8}{1024}\ \text{\&c.}$$

$$2 - \frac{x^2}{2}\Big)\ -\frac{7x^6}{64} - \frac{49}{1024}x^8\ \text{\&c.}$$

$$\times -\frac{7x^6}{128} = -\frac{7x^6}{64} + \frac{7}{256}x^8\ \text{\&c.}$$

$$2\ \Big)\ -\frac{77}{1024}x^8\ \text{\&c.}$$

$$\times -\frac{77x^8}{2048} = -\frac{77}{1024}x^8\ \text{\&c.}$$

Then $\sqrt[4]{1-xx} = 1 - \frac{x^2}{4} - \frac{3x^4}{32} - \frac{7x^6}{128} - \frac{77x^8}{2048}$ &c.

Mult.

$$\begin{array}{rllllll}
\text{Mult.} & 1 & -\frac{x^2}{4} & -\frac{3x^4}{32} & -\frac{7x^6}{128} & -\frac{77x^8}{2048} & \text{\&c.}\\
\text{by} & 1 & -\frac{x^2}{4} & -\frac{3x^4}{32} & -\frac{7x^6}{128} & -\frac{77x^8}{2048} & \text{\&c.}\\
\hline
 & 1 & -\frac{x^2}{4} & -\frac{3x^4}{32} & -\frac{7x^6}{128} & -\frac{77x^8}{2048} & \text{\&c.}\\
 & & -\frac{x^2}{4} & +\frac{x^4}{16} & +\frac{3x^6}{128} & +\frac{7x^8}{512} & \text{\&c.}\\
 & & & -\frac{3x^4}{32} & +\frac{3x^6}{128} & +\frac{9x^8}{1024} & \text{\&c.}\\
 & & & & -\frac{7x^6}{128} & +\frac{7x^8}{512} & \text{\&c.}\\
 & & & & & -\frac{77x^8}{204} & \text{\&c.}\\
\hline
\text{Prod.} & 1 & -\frac{xx}{2} & -\frac{x^4}{8} & -\frac{x^6}{16} & -\frac{5x^8}{128} & \text{\&c.}\\
\text{by} & 1 & -\frac{xx}{2} & -\frac{x^4}{8} & -\frac{x^6}{16} & -\frac{5x^8}{128} & \text{\&c.}\\
\hline
 & 1 & -\frac{xx}{2} & -\frac{x^4}{8} & -\frac{x^6}{16} & -\frac{5x^8}{128} & \text{\&c.}\\
 & & -\frac{xx}{2} & +\frac{x^4}{4} & +\frac{x^6}{16} & +\frac{x^8}{32} & \text{\&c.}\\
 & & & -\frac{x^4}{8} & +\frac{x^6}{16} & +\frac{x^8}{64} & \text{\&c.}\\
 & & & & -\frac{x^6}{16} & +\frac{x^8}{32} & \text{\&c.}\\
 & & & & & -\frac{5x^8}{128} & \text{\&c.}\\
\hline
 & 1 & -xx & .\ 0 & .\ 0 & .\ 0 & \\
\hline
\end{array}$$

Thus by involving the roots themſelves, the original powers are reſtored, which prove the truth of the inveſtigations.

EXAM. 10.

Find the value of $\sqrt{\frac{aa-xx}{cc-xx}} = \frac{\sqrt{aa-xx}}{\sqrt{cc-xx}}$, *in an infinite ſeries.*

1. $\sqrt{aa-xx} = a - \frac{xx}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c. by *Cor. Ex.* 2.

2. $\sqrt{cc-xx} = c - \frac{xx}{2c} - \frac{x^4}{8c^3} - \frac{x^6}{16c^5}$ &c.

now the former divided by the latter, the quotient $= \frac{a}{c} + \frac{axx}{2c^3} - \frac{xx}{2ac} + \frac{3ax^4}{8c^5} - \frac{x^4}{4ac^3} - \frac{x^4}{8a^3c} + \frac{5ax^6}{16c^7} - \frac{3x^6}{16ac^5} - \frac{x^6}{16a^3c^3} - \frac{x^6}{16a^5c}$ &c. that is,

$$= \frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6$$

&c.

Of the Binomial Theorem.

It has already been obſerved, that $\overline{a+b}|^m$, if m be a whole number, ſhews the m^{th} power of $a+b$; but if m be a fraction, as $\frac{1}{2}$, $\frac{1}{3}$, &c. it denotes the ſquare, cube, &c. root of any quantity.

Now $\overline{a+b}|^m = a^m + ma^{m-1}b + \frac{m.m-1}{1.2}a^{m-2}b^2 + \frac{m.m-1.m-2}{1.2.3}a^{m-3}b^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4}a^{m-4}b^4 +$ &c.

Or $\overline{a+b}|^m = a^m + \frac{m}{1} \times \frac{b}{a}A + \frac{n-1}{2} \times \frac{b}{a}$

B +

$\text{B} + \frac{n-2}{3} \times \frac{b}{a}\,\text{C} + \frac{n-3}{4} \times \frac{b}{a}\,\text{D}$ &c. where A, B, C, &c. denote the firſt, ſecond, third, &c. term.

Or $\overline{a+b}|^m = a^m + \frac{ab}{a+b}\,\text{A} + \frac{m+1}{2} \times \frac{b}{a+b}\,\text{B} + \frac{m+2}{3} \times \frac{b}{a+b}\,\text{C} + \frac{m+3}{4} \times \frac{b}{a+b}\,\text{D}$ &c.

And $\overline{a-b}|^m = a^m - ma^{m-1}b + \frac{m.\overline{m-1}}{1\,.\,2}\,a^{m-2}b^2 - \frac{m.\overline{m-1}.\overline{m-2}}{1\,.\,2\,.\,3}\,a^{m-3}b^3 + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{1\,.\,2\,.\,3\,.\,4}\,a^{m-4}b^4 -$ &c.

Or $\overline{a-b}|^m = a^m - \frac{m}{1} \times \frac{b}{a}\,\text{A} - \frac{m-1}{2} \times \frac{b}{a}\,\text{B} - \frac{m-2}{3} \times \frac{b}{a}\,\text{C} - \frac{m-3}{4} \times \frac{b}{a}\,\text{D} -$ &c.

Theſe ſeries are univerſal, and ſerve for raiſing powers, and extraction of roots.

Suppoſe $m = \frac{1}{2}$, then $\overline{a+b}|^m = \sqrt{a+b} = \overline{a+b}|^{\frac{1}{2}}$; and to extract the ſquare root of $a+b$, is no more than to raiſe $a+b$ to the power whoſe exponent or index is $\frac{1}{2}$.

Thus 1. $a^m = a^{\frac{1}{2}}$.

2. $ma^{m-1}b = \frac{1}{2}a^{-\frac{1}{2}}b = \frac{b}{2a^{\frac{1}{2}}}$.

3. $\frac{m.\overline{m-1}}{1\,.\,2}\,a^{m-2}b^2 = \frac{1}{2} \times -\frac{1}{4}a^{-\frac{3}{2}}b^2 = -\frac{b^2}{8a^{\frac{3}{2}}}$.

4. $\frac{m.\overline{m-1}\ \overline{m-2}}{1\,.\,2\,.\,3}\,a^{m-3}b^3 = \frac{1}{2} \times -\frac{1}{4} \times -\frac{1}{2}\,a^{-\frac{5}{2}}b^3 = +\frac{b^3}{16a^{\frac{5}{2}}}$.

Therefore

Therefore $\sqrt{a+b} = a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \frac{b^3}{16a^{\frac{5}{2}}}$ — &c.

Since therefore the power of $\overline{a+b}|^{m} = a^m + \frac{m}{1} a^{m-1}b + \frac{m.m-1}{1\,.\,2} a^{m-2}b^2 + \frac{m.m-1.m-2}{1\,.\,2\,.\,3} a^{m-3}b^3 + \frac{m.m-1.m-2.m-3}{1\,.\,2\,.\,3\,.\,4} a^{m-4}b^4 +$ &c. and conſequently the coefficients or *unciæ* of each term come out as $\frac{m}{1}$, $\frac{m.m-1}{1\,.\,2}$, $\frac{m.m-1.m-2}{1\,.\,2\,.\,3}$, $\frac{m.m-1.m-2.m-3}{1\,.\,2\,.\,3\,.\,4}$, &c. this may eaſily be committed to memory.

Now if we put $P = a$, and $Q = \frac{b}{a}$; then $a^m = P^m$, $\frac{b^2}{a^2} = Q^2$, $\frac{b^3}{a^3} = Q^3$, $\frac{b^4}{a^4} = Q^4$, &c. theſe values being ſubſtituted in the above Theorem, it will be

$$\overline{a+b}|^{m} = P^m + \frac{m}{1} P^m Q + \frac{m.m-1}{1\,.\,2} P^m Q^2 + \frac{m.m-1.m-2}{1\,.\,2\,.\,3} P^m Q^3 + \frac{m.m-1.m-2.m-3}{1\,.\,2\,.\,3\,.\,4} P^m Q^4 \text{ \&c.}$$

And if again we put $P^m = A$

$$\frac{m}{1} P^m Q = B$$

$$\frac{m.m-1}{1\,.\,2} P^m Q^2 = C$$

$$\frac{m.m-1.m-2}{1\,.\,2\,.\,3} P^m Q^3 = D$$

$$\frac{m.m-1.m-2.m-3}{1\,.\,2\,.\,3\,.\,4} P^m Q^4 = E.$$

&c.

Then will $\overline{a+b}|^{m}$ be $= \overline{P+PQ}|^{m} = P^{m} + \frac{m}{1} AQ + \frac{m-1}{2} BQ + \frac{m-2}{3} CQ + \frac{m-3}{4} DQ + \frac{m-3}{5} EQ$ &c.

If we would extract the root of any binomial $a+b$ whoſe index is $\frac{m}{n}$, then inſtead of m in the above general formula there be wrote $\frac{m}{n}$, we ſhall have this general Theorem.

$$\overline{a+b}|^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m-n}{n}} b^{1} + \frac{m.m-n}{1\,.\,2n^{2}} a^{\frac{m-2n}{n}} b^{2} + \frac{m.m-n.m-2n}{1\,.\,2\,.\,3n^{3}} a^{\frac{m-3n}{n}} b^{3} + \frac{m.m-n.m-2n\,m-3n}{1\,.\,2\,.\,3\,.\,4n^{4}} a^{\frac{m-4n}{n}} b^{4} + \&c.$$

or it will be $\overline{a+b}|^{\frac{m}{n}} = \overline{P+PQ}|^{\frac{m}{n}}$

$$= \overset{A}{P^{\frac{m}{n}}} + \overset{B}{\frac{m}{n} AQ} + \overset{C}{\frac{m-n}{2n} BQ} + \overset{D}{\frac{m-2n}{3n} CQ} + \overset{E}{\frac{m-3n}{4n} DQ} + \&c,$$

This is *Sir Iſaac Newton*'s famous Binomial Theorem.

COR. I.

$$\overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} \times :1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m-n}{2n} Q^{2} +$$

$$Q^{2} + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^{3} + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^{4} \text{ \&c.}$$

COR. 2.

$$\frac{1}{\overline{a+b}|^{\frac{m}{n}}} = \overline{a+b}|^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} - \frac{m}{1} \times \frac{b}{a} A - \frac{m+n}{2n} \times \frac{b}{a} B - \frac{m+2n}{3n} \times \frac{b}{a} C - \frac{m+3n}{4n} \times \frac{b}{a} D - \text{\&c.}$$

COR. 3.

$$\sqrt{BC-CC} = B^{\frac{1}{2}}C^{\frac{1}{2}} - \frac{C^{\frac{3}{2}}}{2B^{\frac{1}{2}}} - \frac{C^{\frac{5}{2}}}{8B^{\frac{3}{2}}} - \frac{C^{\frac{7}{2}}}{16B^{\frac{5}{2}}} - \frac{5C^{\frac{9}{2}}}{128B^{\frac{7}{2}}} \text{ \&c.}$$

COR. 4.

$$\overline{AC-AB}|^{\frac{3}{2}} = AC^{\frac{3}{2}} - \frac{3}{2} AC^{\frac{1}{2}} \times AB + \frac{3}{8} \times \frac{AB^{2}}{AC^{\frac{1}{2}}} - \text{\&c.}$$

COR. 5.

$$\overline{AA-BB}|^{\frac{3}{4}} = A^{\frac{3}{2}} - \frac{3B^{2}}{4A^{\frac{1}{2}}} - \frac{3B^{4}}{4.8A^{\frac{5}{2}}} - \frac{3.5B^{6}}{4.8.12A^{\frac{9}{2}}} - \frac{3.5.9B^{8}}{4.8.12.16A^{\frac{13}{2}}} - \text{\&c.}$$

COR.

COR. 6.

$$\frac{\overline{A+B}|^{\frac{3}{2}} - \overline{A-B}|^{\frac{3}{2}}}{3} = \sqrt{A} \times B - \frac{B^3}{4.6AA} -$$

$$\frac{1.3.5B^5}{4.6.8.10A^4} - \frac{1.3.5.7.9B^7}{4.6.8.10.12.14A^6} \text{ \&c.}$$

The uſe of the above binomial Theorem, is to reduce a binomial ſurd into an infinite ſeries; which is done by bare ſubſtitution of the particular letters or quantities in the given expreſſion, inſtead of thoſe in the general Theorem, carefully obſerving the ſigns; the ſame muſt be underſtood of thoſe in the above general Corollaries.

EXAM. I.

To find $\sqrt{aa+xx} = \overline{aa+xx}|^{\frac{1}{2}}$, *in a ſeries, or to extract the ſquare root of* $aa + xx$, *by the Theorem.*

Here $P = aa$, $Q = \frac{xx}{aa}$, $m = 1$, $n = 2$, and

$$P^{\frac{m}{n}} = aa^{\frac{1}{2}} = a = A,$$

$$\frac{m}{n} = \frac{1}{2} \times a \times \frac{x^2}{a^2} = \frac{x^2}{2a} = B,$$

$$\frac{m-n}{2n} = \frac{1-2}{4} = -\frac{1}{4} \times \frac{x^2}{2a} \times \frac{x^2}{a^2} = -\frac{x^4}{8a^3} = C,$$

$$\frac{m-2n}{3n} = \frac{1-4}{6} = -\frac{3}{6} \times -\frac{x^4}{8a^3} \times \frac{x^2}{a^2} =$$

$$\frac{3x^6}{48a^5} = \frac{x^6}{16a^5} = D,$$

$$\frac{m-3n}{4n} = \frac{1-6}{8} = -\frac{5}{8} \times \frac{x^6}{16a^5} \times \frac{x^2}{a^2} = -$$

$$\frac{5x^8}{128a^7} = E,$$

$m - 4n$

$\frac{m-4n}{5n} = \frac{1-8}{10} = -\frac{7}{10} \times - \frac{5x^8}{128a^7} \times \frac{x^2}{a^2} =$
$\frac{7x^{10}}{256a^9} = F,$

$\frac{m-5n}{6n} = \frac{1-10}{12} = -\frac{9}{12} = -\frac{3}{4} \times \frac{7x^{10}}{256a^9}$
$\times \frac{x^2}{a^2} = -\frac{21x^{12}}{1024a^{11}} = G,$

Therefore $\sqrt{aa+xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ $- \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} - \frac{21x^{12}}{1024a^{11}}$ &c. the ſame as *Ex.* 2. extract roots.

EXAM. 2.

To find the value of $\sqrt{x-xx}$ *in a ſeries.*

Here $P = x$, $Q = -\frac{xx}{x} = -x$, $m = 1$, $n = 2$;

then $P^{\frac{m}{n}} = x^{\frac{1}{2}}$
$\frac{m}{n} = \frac{1}{2} \times x^{\frac{1}{2}} \times -x = -\overline{x^3}|^{\frac{1}{2}} \times \frac{1}{2} = -\frac{1}{2}x^{\frac{3}{2}}$
$\frac{m-n}{2n} = \frac{1-2}{4} = -\frac{1}{4} \times -\overline{x^3}|^{\frac{1}{2}} \times -x = -$
$\overline{\tfrac{1}{4}x^3}|^{\frac{1}{2}} \times -\frac{1}{2} = \frac{1}{8}x^{\frac{5}{2}}.$

Therefore $\sqrt{x-xx} = x^{\frac{1}{2}} - \tfrac{1}{2}x^{\frac{3}{2}} + \tfrac{1}{8}x^{\frac{5}{2}} + \tfrac{1}{16}x^{\frac{7}{2}}$ $- \tfrac{5}{128}x^{\frac{9}{2}}$ &c.

EXAM. 3.

Extract the ſquare root of $aa - xx$, *or* $\overline{aa-xx}|^{\frac{1}{2}}$ *in a ſeries.*

Here $P = aa$, $Q = -\frac{xx}{aa}$, $m = 1$, $n = 2$, $P^{\frac{m}{n}} = aa^{\frac{1}{2}} = a = A$.

$$\frac{m}{n} = \frac{1}{2} \times a \times -\frac{xx}{aa} = -\frac{ax^2}{2a^2} = -\frac{x^2}{2a} = B.$$

$$\frac{m-n}{2n} = \frac{1-2}{4} = \frac{-1}{4} \times \frac{-x^2}{2a} \times \frac{-x^2}{aa} = -\frac{x^4}{8a^3} = C.$$

$$\frac{m-2n}{3n} = \frac{1-4}{6} = \frac{-3}{6} \times -\frac{x^4}{8a^3} \times -\frac{x^2}{aa} = -\frac{3x^6}{6.8a^5} = -\frac{x^6}{16a^5} = D.$$

$$\frac{m-3n}{4n} = \frac{1-6}{8} = \frac{-5}{8} \times \frac{-x^6}{16a^5} \times \frac{-xx}{aa} = -\frac{5x^8}{128a^7} \ \&c.$$

Therefore $\sqrt{aa-xx} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

EXAM. 4.

Expreſs $\frac{3}{\sqrt[3]{a+x}}$ *in an infinite ſeries.*

Now $\frac{3}{\sqrt[3]{a+x}} = \overline{a+x}|^{-\frac{3}{1}} = \overline{a+x}|^{-3}$.

Here $P = a$, $Q = \frac{x}{a}$, $m = -3$, $n = 1$; then

$$P^{\frac{m}{n}} = a^{-\frac{3}{1}} = \frac{1}{a^3} = A.$$

$$\frac{m}{n} = \frac{-3}{1} \times \frac{1}{a^3} \times \frac{x}{a} = \frac{-3x}{a^4} = B.$$

$m - n$

$$\frac{m-n}{2n} = \frac{-3-1}{2} = \frac{-4}{2} \times \frac{-3x}{a^4} \times \frac{x}{a} = \frac{12x^2}{2a^5} = \frac{6x^2}{a^5} = C.$$

$$\frac{m-2n}{3n} = \frac{-3-2}{3} = \frac{-5}{3} \times \frac{6x^2}{a^5} \times \frac{x}{a} = -\frac{30x^3}{3a^6} = -\frac{10x^3}{a^6} = D.$$

$$\frac{m-3n}{4n} = \frac{-3-3}{4} = \frac{-6}{4} = \frac{-3}{2} \times \frac{-10x^3}{a^6} \times \frac{x}{a} = \frac{30x^4}{2a^7} = \frac{15x^4}{a^7}$$

$$\frac{m-4n}{5n} = \frac{-3-4}{5} = \frac{-7}{5} \times \frac{15x^4}{a^7} \times \frac{x}{a} = -\frac{105x^5}{5a^8} = -\frac{21x^5}{a^8} = E.$$

Therefore $\frac{3}{\sqrt[3]{a+x}} = 3 \times : \frac{1}{a^3} - \frac{3x}{a^4} + \frac{6x^2}{a^5} - \frac{10x^3}{a^6} + \frac{15x^4}{a^7} - \frac{21x^5}{a^8}$ &c.

EXAM. 5.

Extract the cube root of $a^3 + x^3$.

Now $\sqrt[3]{a^3 + x^3} = \overline{a^3 + x^3}|^{\frac{1}{3}}$.

Here $P = a^3$; $Q = \frac{x^3}{a^3}$; $m = 1$; $n = 3$; then

$$P^{\frac{m}{n}} = \overline{a^3}|^{\frac{1}{3}} = a = A.$$

$$\frac{m}{n} = \frac{1}{3} \times a \times \frac{x^3}{a^3} = \frac{ax^3}{3a^3} = \frac{x^3}{3a^2} = B.$$

$$\frac{m-n}{2n} = \frac{1-3}{6} = \frac{-2}{6} = \frac{-1}{3} \times \frac{x^3}{3a^2} \times \frac{x^3}{a^3} = -\frac{x^6}{9a^5} = C.$$

$$\frac{m-2n}{3n} = \frac{1-6}{9} = \frac{-5}{9} \times \frac{-x^6}{9a^5} \times \frac{x^3}{a^3} = \frac{5x^9}{81a^8} = \text{D}.$$

$$\frac{m-3n}{4n} = \frac{1-9}{12} = \frac{-8}{12} = \frac{-2}{3} \times \frac{5x^9}{81a^8} \times \frac{x^3}{a^3}$$

$$= -\frac{10x^{12}}{243a^{11}} = \text{E}.$$

Therefore $\sqrt[3]{a^3+x^3} = a + \frac{x^3}{a^3} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$ $- \frac{10x^{12}}{243a^{11}}$ &c.

EXAM. 6.

Find the value of $\sqrt[5]{a^5+a^4x-x^5}$ *or* $\overline{a^5+a^4x-x^5}|^{\frac{1}{5}}$ *in a ſeries.*

Here $\text{P} = a^5$; $\text{Q} = \frac{a^4x - x^5}{a^5}$; $m = 1$; $n = 5$;

then $\text{P}^{\frac{m}{n}} = \overline{a^5}|^{\frac{1}{5}} = a = \text{A}$.

$$\frac{m}{n} = \frac{1}{5} \times a \times \frac{a^4x-x^5}{a^5} = \frac{a^5x-ax^5}{5a^5} = \frac{a^4x-x^5}{5a^4} = \text{B}.$$

$$\frac{m-n}{2n} = \frac{1-5}{10} = \frac{-4}{10} = \frac{-2}{5} \times \frac{a^4x-x^5}{5a^4} \times$$

$$\frac{a^4x-x^5}{a^5} = -\frac{2a^8x^2+4a^4x^6-2x^{10}}{25a^9}.$$

Therefore $\sqrt[5]{a^5+a^4x-x^5} = a + \frac{a^4x-x^5}{5a^4} -$ $\frac{2a^8x^2+4a^4x^6-2x^{10}}{25a^9}$ &c.

EXAM. 7.

Extract the cube root of $\overline{a+x}|^4$.

Now $\overline{\overline{a+x}|^4}|^{\frac{1}{3}} = \overline{a+x}|^{\frac{4}{3}}$.

Here $\text{P} = a$; $\text{Q} = \frac{x}{a}$; $m = 4$; $n = 3$; then

$$P^{\frac{m}{n}} = a^{\frac{4}{3}} = A.$$

$$\frac{m}{n} = \frac{4}{3} \times \overline{a^4}\big|^{\frac{1}{3}} \times \frac{x}{a} = \frac{4}{3} \times \overline{\frac{x^3a^4}{a^3}}\Big|^{\frac{1}{3}} = \frac{4}{3} \times$$

$$\overline{xa}\big|^{\frac{1}{3}} = B.$$

$$\frac{m - n}{2n} = \frac{4 - 3}{6} = \frac{1}{6} \times \frac{4}{3} = \frac{2}{9} \times \overline{\frac{x^3a^3}{a^3}}\Big|^{\frac{1}{3}} \times$$

$$\frac{x}{a} = \frac{2}{9} \times \overline{\frac{x^6a^4}{a^6}}\Big|^{\frac{1}{3}} = \frac{2x^2}{9a^{\frac{2}{3}}} = C.$$

$$\frac{m - 2n}{3n} = \frac{4 - 6}{9} = \frac{-2}{9} \times \frac{2}{9} = \frac{-4}{81} \times \overline{\frac{x^6a^4}{a^6}}\Big|^{\frac{1}{3}}$$

$$\times \frac{x}{a} = \overline{\frac{x^9a^4}{a^9}}\Big|^{\frac{1}{3}} \times \frac{-4}{81} = \frac{-4x^3}{81a^{\frac{5}{3}}} = D.$$

$$\frac{m - 3n}{4n} = \frac{4 - 9}{12} = \frac{-5}{12} \times \frac{-4}{81} = \frac{20}{972} = \frac{5}{243} \times$$

$$\overline{\frac{x^9a^4}{a^9}}\Big|^{\frac{1}{3}} \times \frac{x}{a} = \overline{\frac{x^{12}a^4}{a^{12}}}\Big|^{\frac{1}{3}} \times \frac{5}{243} = \frac{5x^4}{243a^{\frac{8}{3}}} = E.$$

Therefore $\sqrt[3]{\overline{a + x^4}} = a^{\frac{4}{3}} + \frac{4}{3} \times \overline{xa}\big|^{\frac{1}{3}} + \frac{2x^2}{9a^{\frac{2}{3}}} -$

$\frac{4x^3}{81a^{\frac{5}{3}}} + \frac{5x^4}{243a^{\frac{8}{3}}}$ &c.

EXAM. 8.

Find the value of $\frac{a}{\sqrt[3]{x^3 - a^2x}}$, or $a \times \overline{x^3 - a^2x}\big|^{-\frac{1}{3}}$, *in a ſeries.*

Here $P = x^3$; $Q = \frac{-a^2x}{x^3} = \frac{-a^2}{xx}$; $m = -1$;

$n = 3$; then $P^{\frac{m}{n}} = \overline{x^3}\big|^{-\frac{1}{3}} = x^{-1} = A.$

$$\frac{m}{n} = \frac{-1}{3} \times \frac{1}{x} \times \frac{-a^2}{x^2} = \frac{a^2}{3x^3} = B.$$

m—n

$$\frac{m-n}{2n}=\frac{-1-3}{6}=\frac{-4}{6}=\frac{-2}{3}\times\frac{a^2}{3x^3}\times\frac{-a^2}{x^2}$$

$$=\frac{2a^4}{9x^5}=\text{C.}$$

$$\frac{m-2n}{3n}=\frac{-1-6}{9}=\frac{-7}{9}\times\frac{2a^4}{9x^5}\times\frac{-a^2}{x^2}=\frac{14a^6}{81x^7}$$

$$=\text{D.}$$

$$\frac{m-3n}{4n}=\frac{-1-9}{12}=\frac{-10}{12}=\frac{-5}{6}\times\frac{14a^6}{81a^7}\times$$

$$\frac{-a^2}{x^2}=\frac{70a^8}{486x^9}=\frac{35a^8}{243x^9}=\text{E.}$$

Then $\frac{a}{\sqrt[3]{a^3-a^2x}}=a\times:\frac{1}{x}+\frac{a^2}{3x^3}+\frac{2a^4}{9x^5}+\frac{14a^6}{81x^7}+\frac{35a^8}{243x^9}$ &c.

EXAM. 9.

Find the value of $\frac{1}{\sqrt{2rx-xx}}$ *or of* $\overline{2rx-xx}|^{-\frac{1}{2}}$, *in a series.*

Here $P=2rx$; $Q=\frac{-xx}{2rx}=\frac{-x}{2r}$; $m=-1$, $n=2$; then $P^{\frac{m}{n}}=\overline{2rx}|^{\frac{1}{2}}=\frac{1}{\sqrt{2rx}}=\text{A}$.

$$\frac{m}{n}=\frac{-1}{2}\times\frac{1}{\sqrt{2rx}}=\frac{-1}{2\sqrt{2rx}}\times\frac{-x}{2r}=\frac{x}{4r\sqrt{2rx}}$$

$$=\text{B.}$$

$$\frac{m-n}{2n}=\frac{-1-2}{4}=\frac{-3}{4}\times\frac{x}{4r\sqrt{2rx}}=\frac{-3x}{4.4r\sqrt{2rx}}$$

$$\times\frac{-x}{2r}=\frac{3xx}{2.4.4r^2\sqrt{2rx}}=\text{C.}$$

$$\frac{m-2n}{3n}=\frac{-1-4}{6}=\frac{-5}{6}\times\frac{3xx}{2.4.4r^2\sqrt{2rx}}=$$

$$\frac{-3.5xx}{2.4.4.6r^2\sqrt{2rx}} \times \frac{-x}{2r} = \frac{3.5x^3}{2.6.32r^3\sqrt{2rx}} = D.$$

$$\frac{m-3n}{4n} = \frac{-1-6}{8} = \frac{-7}{8} \times \frac{3.5x^3}{2.6.32r^3\sqrt{2rx}} \times \frac{-x}{2r}$$

$$= \frac{3.5.7x^4}{2.2.6.8\ 32r^4\sqrt{2rx}} = E.$$

Therefore $\frac{1}{\sqrt{2rx-xx}} = \frac{1}{\sqrt{2rx}} + \frac{x}{4r\sqrt{2rx}} +$ $\frac{3xx}{2.4.4r^2\sqrt{2rx}} + \frac{3.5x^3}{2.6.32r^3\sqrt{2rx}} +$ $\frac{3.5.7x^4}{2.2.6.8.32r^4\sqrt{2rx}}$ &c. that is,

$$\frac{1}{\sqrt{2rx-xx}} = \frac{1}{\sqrt{2rx}} \times : 1 + \frac{x}{4r} + \frac{3x^2}{32rr} +$$
$$\frac{3.5x^3}{2.6.32r^3} + \frac{3.5.7x^4}{2.2.6.8.32r^4}, \text{ \&c.}$$

EXAM. 10.

Extract the biquadrate root of $1 - xx$, *or to put* $\sqrt[4]{1-xx} = \overline{1-xx}^{\frac{1}{4}}$ *into a ſeries.*

Here $P = 1$, $Q = \frac{-xx}{1} = -xx$, $m = 1$, $n = 4$; and $P^{\frac{m}{n}} = 1^{\frac{1}{4}} = 1$.

$$\frac{m}{n} = \frac{1}{4} \times 1 \times -xx = -\frac{xx}{4}$$

$$\frac{m-n}{2n} = \frac{1-4}{8} = \frac{-3}{8} \times -\frac{xx}{4} \times -\frac{xx}{1} = -\frac{3x^4}{4.8}$$

$$\frac{m-2n}{3n} = \frac{1-8}{12} = \frac{-7}{12} \times -\frac{3x^4}{4.8} \times -xx = -$$
$$\frac{3.7x^6}{4.8.12}$$

$$\frac{m-3n}{4n} = \frac{1-12}{16} = \frac{-11}{16} \times -\frac{3.7x^6}{4.8.12} = \frac{3.7.11x^6}{4.8.12.16}$$
$$\times -xx = -\frac{3.7.11x^8}{4.8.12.16} = -\frac{77x^8}{2048}.$$

Therefore $\sqrt[4]{1-xx} = 1 - \frac{x^2}{4} - \frac{3x^4}{4.8} - \frac{3.7x^6}{4.8.12} - \frac{7.11x^8}{4.4.8.16}$ &c. See *Ex.* 9. extraction of roots.

EXAM. 11.

Put $\sqrt[5]{1-xx} = \overline{1-xx}|^{\frac{1}{5}}$ *into an infinite ſeries.*

Here P = 1, Q = $-xx$, $m = 1$, $n = 5$, and $P^{\frac{m}{n}} = 1$.

$$\frac{m}{n} = \frac{1}{5} \times 1 \times -xx = -\frac{xx}{5}$$

$$\frac{m-n}{2n} = \frac{1-5}{10} = \frac{-4}{10} = \frac{-2}{5} \times -\frac{xx}{5} \times -xx = -\frac{2x^4}{25}$$

$$\frac{m-2n}{3n} = \frac{1-10}{15} = \frac{-9}{15} = -\frac{3}{5} \times -\frac{2x^4}{25} \times -xx = -\frac{6x^6}{125}$$

$$\frac{m-3n}{4n} = \frac{1-15}{20} = -\frac{14}{20} = -\frac{7}{10} \times -\frac{6x^6}{125} \times -xx = -\frac{21x^8}{625}.$$

Therefore $\overline{1-xx}|^{\frac{1}{5}} = 1 - \frac{x^2}{5} - \frac{2x^4}{25} - \frac{6x^6}{125} - \frac{21x^8}{625}$ — &c.

EXAM. 12.

Find $\sqrt{r^2x - sx^2}$ *in a ſeries.*

Here $P = r^2x$, $Q = -\frac{sx^2}{r^2x} = -\frac{sx}{r^2}$, $m = 1$, $n = 2$, and $P^{\frac{m}{n}} = rx^{\frac{1}{2}}$.

$$\frac{1}{2} \times rx^{\frac{1}{2}} \times -\frac{sx}{r} = -\frac{rsx^{\frac{3}{2}}}{2r^2} = -\frac{sx^{\frac{3}{2}}}{2r}$$

$$-\frac{1}{4} \times -\frac{sx^{\frac{3}{2}}}{2r} \times -\frac{sx}{r^2} = -\frac{s^2x^{\frac{5}{2}}}{4.2r^3}$$

$$-\frac{3}{6} \times -\frac{s^2x^{\frac{5}{2}}}{4.2r^3} \times -\frac{sx}{r^2} = -\frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5}$$

$$-\frac{5}{8} \times -\frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5} \times -\frac{sx}{r^2} = -\frac{3.5s^4x^{\frac{9}{2}}}{2.4.6.8r^7}.$$

Therefore $\sqrt{r^2x - sx^2} = rx^{\frac{1}{2}} - \frac{sx^{\frac{3}{2}}}{2r} - \frac{s^2x^{\frac{5}{2}}}{4.2r^3} - \frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5} - \frac{3.5s^4x^{\frac{9}{2}}}{2.4.6.8r^7}$ &c.

Corollary.

By the ſame Theorem $\overline{1+z}|^{1+z} = 1 + z + z^2 + \frac{z^3}{2} + \frac{z^4}{3} + \frac{z^5}{12} + \frac{3z^6}{40}$ &c. and $\overline{1-x}|^{1+z} = 1 - x - zx + \frac{zx^2 + z^2x^2}{2} + \frac{zx^3 - z^3x^3}{6} + \frac{2zx^4 - z^2x^4 - 2z^3x^4 + z^4x^4}{24}$ &c. $= 1 - x - xz + \frac{\overline{z+zz}.x^2}{2} + \frac{\overline{z-z^3}.x^3}{6} + \frac{\overline{2z-z^2-2z^3+z^4} \times x^4}{24}$.

And $\overline{1-x}|^{2-z} = 1 - 2x + zx + \frac{\overline{2-3z+zz}.x^2}{2} + \frac{\overline{2z-3z^2+z^3}\,x^3}{0}$ &c. Theſe ſeries are uſeful in ſolving exponential Problems.

We can alſo obtain a *Formula* for the power of an infinitonomial.

Let the ſeries $a + bz + cz^2 + dz^3 + ez^4$ &c. be raiſed to the p power, then will

$$a^p + pa^{p-1}bz + pa^{p-1}cz^2 + pa^{p-1}dz^3$$

$$+ p \times \frac{p-1}{2} a^{p-1}b^2z^2 + p \times \frac{p-1}{2} a^{p-2} \times 2bcz^3$$

$$+ p \times \frac{p-1}{2} \times \frac{p-2}{3} a^{p-3}b^3z^3$$

&c.

Let the root $\overline{a+b}|^{\frac{m}{p}} = a + bz + cz^2 + dz^3$ &c.

Now $\overline{a+b}|^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} a^{m-2}$ $b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3$ &c. and if they be compared with the terms of the power of the Infinitonomial $a + bz + cz^2 + dz^3$ &c. raiſed to the p power, we ſhall have

$$a^p = a^m$$

$$pa^{p-1}bz = ma^{m-1}b,$$

$$pa^{p-1}cz^2 + p \times \frac{p-1}{2} a^{p-2}b^2z^2 = m \times \frac{m-1}{2} a^{m-2}b,$$

$$pa^{p-1}dz^3 + p \times \frac{p-1}{2} a^{p-2} \times 2bcz^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} a^{p-3}b^3z^3 = m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 \text{ \&c.}$$

Whence $a = a^{\frac{m}{p}}$,

$$bz = \frac{m}{p} \times \frac{a^{m-1}b}{a^{m-\frac{m}{p}}} = \frac{m}{p} a^{\frac{m-p}{p}} b,$$

$$cz^2 = \frac{m.m-p}{1.2p^2} a^{\frac{m-2p}{p}} b^2,$$

&c.

I ſhall not purſue this ſubject any further, but only ſet down ſome Theorems for reverting of ſeries deduced from Prob. 62. of Mr. *Emerſon*'s Algebra.

When equations of this form, $az + bz^2 + cz^3 + dz^4$ &c. $= x$, happen to ariſe in the ſolution of a Problem, the value of z cannot be obtained but by reverſion of ſeries.

EXAM. 1.

Given $az + bz^2 + cz^3 + dz^4 + ez^5$ *&c.* $= x$ *a given number, then,*

THEOREM I.

$$z = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{\overline{2bb - ac}}{a^5}x^3 + \frac{\overline{5abc - a^2d - 5b^3}}{a^7}x^4 + \frac{\overline{14b^4 - 21ab^2c + 6a^2bd + 3a^2c^2 - a^3e}}{a^9}x^5 \text{ \&c.}$$

EXAM. 2.

If $z + bz^2 + cz^3 + dz^4$ &c. $= x$, *then,*

THEOREM II.

$$z = x - bx^2 + \overline{2bb - c} \times x^3 + \overline{5bc - 5b^3 - d} \times x^4 \text{ \&c.}$$

EXAM. 3.

If $az + bz^3 + cz^5 + dz^7$ &c. $= x$, *then,*

THEOREM III.

$$z = \frac{x}{a} - \frac{bx^3}{a^4} + \frac{\overline{3bb - ac}}{a^7} \times x^5 + \frac{\overline{8abc - a^2d - 12b^3}}{a^{10}} \times x^7 \text{ \&c.}$$

EXAM. 4.

If $z + bz^3 + cz^5 + dz^7$ *&c.* $= x$, *then*

THEOREM IV.

$$z = x - bx^3 + \overline{3bb - c} \times x^5 + \overline{8bc - 12b^3 - d} \times x^7 \text{ \&c.}$$

EXAM.

EXAM. 5.

If $z^2 + bz^3 + cz^4 + dz^5$ *&c.* $= x$, *then,*

THEOREM V.

$$z = \sqrt{x} - \frac{bx}{2} + \frac{\overline{5bb - 4c}}{8} x^{\frac{3}{2}} + \frac{\overline{3bc - 2b^3 - d}}{2} x^{\frac{5}{2}} \text{ \&c.}$$

EXAM. 6.

If $z^2 + bz^4 + cz^6 + dz^8$ *&c.* $= 3x$, *then,*

THEOREM VI.

$$z = \sqrt{3x} - \frac{b}{2} \times \overline{3x}|^{\frac{3}{2}} + \frac{\overline{7bb - 4c}}{8} \times \overline{3x}|^{\frac{5}{2}} +$$
$$\frac{\overline{36bc - 33b^3 - 8d}}{16} \times \overline{3x}|^{\frac{7}{2}} \text{ \&c.}$$

EXAM. 1.

Given $\frac{2}{3} x^2 + \frac{1}{3} x^4 + \frac{5}{8} x^6 + \frac{3}{4} x^8$ *&c.* $= a$, *to find the value of* x.

1. Divide by $\frac{2}{3}$, and it is $x^2 + \frac{1}{2} x^4 + \frac{15}{16} x^6 + \frac{9}{8} x^8$ &c. $= \frac{3a}{2}$.

Or by ſubſtitution, $x^2 + bx^4 + cx^6 + dx^8$ &c. $=$ A. Now by *Theo.* 6. we have $x = \sqrt{A} - \frac{b}{2} A^{\frac{3}{2}} + \frac{\overline{7bb - 4c}}{8} A^{\frac{5}{2}} + \frac{36bc - 33b^3 - 8d}{16} A^{\frac{7}{2}}$ &c.

EXAM. 2.

Let $m - px = n - x - \frac{1}{6} x^3 - \frac{1}{24} x^5 - \frac{1}{83} x^7 - \frac{1}{262} x^9$ *&c. to find the value of* x, *or to revert the ſeries.*

By tranſpoſition.

$$n - m = \overline{1 - px} + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{83}x^7 + \frac{1}{262}x^9 \text{ \&c.}$$

Put $n - m = A$, and $1 - p = a$, and b, c, d, e, for the known coefficients of the powers of x; then $ax + bx^3 + cx^5 + dx^7 + ex^9$ &c. $= A$.

By Theo. III.

$$x = \frac{A}{a} - \frac{bA^3}{a^4} + \frac{\overline{3b^2 - ac}}{a^7}A^5 + \frac{\overline{8abc - a^2d - 12b^3}}{a^{10}}A^7 + \frac{\overline{55b^4 - 55ab^2c + 10a^2bc + 5a^2c^2 - a^3e}}{a^{13}}A^9.$$

The reducing of fractional quantities and ſurds into an infinite ſeries by diviſion, extraction of roots, or by the binomial Theorem, is of the greateſt uſe in ſolving many abſtruſe Problems in Fluxions, as alſo the method of reverſion of ſeries, or extracting the root of an infinite equation; as they ſo often occur in finding the length of curve lines, &c. So that I would recommend to every one who is deſirous to be more than ſuperficially verſed in the Mathematics, to make himſelf maſter of them.

SECT. III.

Of FLUXIONS,

I.

General Definition.

If we conſider quantities as generated by continual motion, it is evident, that in equal ſpaces of times they will become greater, or leſs proportionally, as the *velocity* of the motion, by which they are ſo generated, is greater or leſs; and therefore the velocity of the motion is very properly called *Fluxion*, and the quantity generated *Fluent*. For Fluxions are not magnitudes, but the *velocities* with which magnitudes, varying by a continued motion, increaſe or decreaſe. And this is agreeable to the noble inventor *Sir Iſaac Newton*, who ſays, "I conſider mathematical quantities here, not as conſiſting of very ſmall parts, but as deſcribed by a continued motion. Lines are deſcribed, and thereby generated, not by the appoſition of parts, but by the continued motion of points; ſuperficies by the motion of lines, &c. therefore I conſider quantities which increaſe in equal times, and by increaſing are generated. I ſought a method of defining quantities from the velocities of the motions or increments with which they are generated, and I call theſe velocities of the motions or increments *Fluxions*, and the generated quantity *Fluents*." Therefore it is plain, that magnitudes are to be conſidered, Fig. ed, not under the notion of being increaſed by a repeated acceſſion of parts, but as generated by a continued local motion, or flux of their elementary parts; and the quantities thus generated are ſaid to *flow*, and called *fluents*, or *flowing quantities*, and ſometimes *variable*, or *indeterminate* quantities; becauſe they are capable of receiving of an indefinite number of particular values, in a regular order of ſucceſſion. Therefore any quantity may be ſuppoſed to be generated by continual increaſe, after the ſame manner that ſpace is deſcribed by local motion.

Hence in the method of Fluxions, geometrical magnitudes are not preſented to the mind as compleatly formed at once, but as riſing gradually before the imagination, by the motion of ſome of their extremes.

1. Thus the line AB may be conceived to be traced out gradually by a point moving along from A to B, either with an equable motion, or with a velocity in any manner varied. And the velocity, or degree of ſwiftneſs with which this point moves in any part of the line AB deſcribed, or at any moment of the time of deſcription, is called the *fluxion* of the line at that place, or at that moment of
2. time. Or, if we ſuppoſe with Mr. *Robins*, there is an unbounded ſpace ACB formed by two lines AB, AC, and there is another line DE erected on AB; now if this line DE be put in motion (ſo as to move always parallel to itſelf) as ſoon as it has paſſed the point A, there will be a ſpace bounded on all ſides between thoſe three lines, for when DE has flowed or moved to FG, they will include the ſpace AGF; it is evident, that this ſpace will increaſe faſter or ſlower, according to the degree of velocity DE ſhall move with; for though DE move with an equable motion, the ſpace AGF would not for that reaſon increaſe equably; but when the line

AC was fartheſt diſtant from AB, the ſpace AFH Fig. would increaſe faſteſt; now the velocity wherewith 2. the ſpace at all times increaſes, is called the *fluxion* of that ſpace.

Or thus,

Art. 1. Indeterminate quantities, which continually increaſe or decreaſe, are called *variable*, or *flowing quantities*; but thoſe which always remain the ſame, whilſt others increaſe or decreaſe, are called conſtant, or determinate quantities.

2. Suppoſe the ordinates BC, BD, ſo move along 3. the baſe AF, with a motion always parallel to themſelves, that the ordinate BD always remaining the ſame, the point D may deſcribe the right line GD*d*; and in the mean time while the ordinate BC continually increaſes or decreaſes, the point C may deſcribe the curve AC*c*, then the abſciſſa AB, ordinate BC, curve A*c*, and the areas ACB, AGDB, are indeterminate, or flowing quantities, but the right line BD is a conſtant quantity.

3. The flowing quantities, as AB, BC, increaſing in equal times, with a greater or leſs velocity by which they increaſe, become greater or leſs as they are generated; for if the point B always move with a ſwifter velocity than the point C in the line BC, the increments B*b* of the fluent AB, will be greater than the increments E*c* of the fluent BC, generated in the ſame time. The velocities with which thoſe increments B*b*, E*c*, generated in the ſame time, firſt begin to be, namely, till *bc* coincide with BC, are called *Fluxions*.

4. The method for finding *Fluxions* of flowing, or determinate quantities, is called *the direct method of Fluxions*; and the method for finding *Fluents* from Fluxions is called, *the inverſe method of Fluxions*.

5. The velocities, by which the increments of flowing quantities, generated in the ſame time, firſt begin to be, are uniform.

Fig. 3.

DEMONSTRATION.

Since the curve AC*c* can be deſcribed by the motion of the point C, moving with any finite velocity, if the velocity of that point, in the direction of CE, parallel to AB, be ſuppoſed uniform; the velocity of the ſame point in the direction of E*c*, according to the various nature of the curve AC*c*, will indeed be various in different parts of the curve, namely, in C and *c*; but by how much the nearer the point *c* ſhall approach to C, ſo much leſs will the variation of the velocity in the points C and *c* be in the direction of E*c*; ſo that when the point *c* coincides with C, the variation of the velocity through E*c* ceaſes. Wherefore the velocities with which the increments of flowing quantities, generated in the ſame time firſt begin to be, are uniform. Q. E. D.

6. Therefore when the uniform velocities are proportional to the ſpaces gone over in the ſame time, it is manifeſt (Art. 1.) that Fluxions are, in the ratio of the increments, generated in the ſame time, while they firſt begin to be, or ultimately vaniſh; therefore to find the relation of Fluxions, we muſt take the increments of the fluents generated in the ſame time, and to conſider the prime ratio of their naſcent increments, or the ultimate ratio of their evaneſcent, as the relation of the Fluxions.

7. Hence the ſum of the Fluxions is as the ſum of the naſcent or evaneſcent increments, but the ſum of all the naſcent increments is the flowing quantity itſelf; for if the whole area A*cb* be ſuppoſed to be divided into parallelograms as BE, and the number of them to be increaſed, and the breadth B*b* be diminiſhed *ad infinitum*, the ſum of all the naſcent increments B*b*, from A to *b*, will be the fluent A*b*; and the ſum of all the increments E*c*, from A to *c*, will be the fluent *bc*; and the ſum of

all

Fig. 3. all the C*c*, will be the fluent arch A*c*; and the ſum of all the parallelograms BE, the fluent area A*cb*; (by Lem. 2. *Principia*, Lib. 1.) therefore the ſum of the Fluxions is as the flowing quantity.

8. Since any Fluxion, either of the abſciſſa AB, or of the ordinate CB, or of the arch AC, may be looked upon at pleaſure as uniform; it appears, (Art. 5.) that the firſt Fluxion of the ſecond, third, &c. Fluxions, may be conſidered as conſtant, and may be expounded by a finite conſtant quantity.

9. Wherefore, when variable quantities have their Fluxions, which again may be variable, it is certain there are Fluxions of Fluxions, or rather, various infinite orders of Fluxions; and the Fluxions of finite fluents are called *firſt Fluxions*, thoſe of the ſecond, *ſecond Fluxions*, &c.

10. Draw the right line VTH, which let it touch the curve in C, and meet *bc* and BA produced, in T and V; let *bc* return into its former place BC, and the ultimate form of theſe evaneſcent triangles CE*c* and CET is ſimilar, and their ultimate ratio that of equality, (by Lem. 8. *Principia*.) therefore the firſt Fluxions of AB, BC, and AC, are as CE, ET, and CT, the ſides of the triangle CET, and may therefore be expounded by thoſe ſides, or, which is the ſame thing, by the ſides VB, CB, and VC, of the triangle VBC, which is ſimilar to the triangle CET.

11. Since the areas B*bc*C and B*bd*D are deſcribed in the ſame time by the uniform motion of the ordinates BC and BD, thoſe naſcent or evaneſcent areas will be as the Fluxions of the areas ACB and ABDG, (Art. 6.) but the naſcent area B*bc*C does not differ from the parallelogram BE, (Lem. 3. *Principia*); therefore the Fluxions of the areas ACB and ABDG are in the prime ratio of the naſcent parallelograms BE, B*d*, or (becauſe the ſide *bd* is common) in the ratio of the ordinates BC, BD.

12. If

Fig. 3. 12. If about the centre B, a circle be deſcribed with the radius BC, and carried with an uniform motion along the abſciſſa AB, and at right angles to it, it will deſcribe the ſame ſolid which would be generated by the revolution of ACB about the axis AB, and the Fluxion of the generated ſolid ABC will be as the product of that circle drawn into B*b*, the naſcent increment of the abſciſſa AB; and the Fluxion of its ſuperficies of that generated ſolid will be as the product of the perimeter of the ſame circle into the arch C*c*, or the naſcent tangent CT; that is, as the circumference of the generating circle, and the Fluxion of the curve conjunctly.

DEMONSTRATION.

The naſcent parallelogram BE, does not differ from the naſcent figure B*bc*C (by Lem. 2. *Principia*, Lib. 1.) and therefore the naſcent increment of the ſolid, generated by the revolution of the parallelogram BE about the ſide B*b*; and this ſolid is a cylinder equal to the product of the area of a circle deſcribed with the radius CB, into the altitude B*b*; therefore the naſcent increment of the ſolid, generated by the motion of the circle CB about the axis AB, and therefore its Fluxion is as the rectangle of the area of the circle into B*b*, the naſcent increment of the abſciſſa AB. Likewiſe when the naſcent arch C*c* coincides with the tangent CT (Lem. 7. *Princ.*) the naſcent ſuperficies, generated by the revolution of the figure B*bc*C, is equal to the ſuperficies of a fruſtum of a cone, and therefore equal to the product of half the ſum of the peripheries, whoſe radii are BC, *bc*, into the ſide CT; or becauſe *bc* = BC, (by Lem. 2. *Princ.*) equal to the product of the periphery of the circle whoſe radius is BC, into the ſide CT, or naſcent arch C*c*; therefore that product is the naſcent increment of the ſuperficies of the curve deſcribed by the revolution

lution of AC, and conſequently is as the Fluxion of that ſuperficies, (Art. 6.) Q. E. D. Fig. 3.

13. By the like arguments, we can find Fluxions from fluents expreſſed by calculation, by inſtituting an analyſis in finite quantities, and inveſtigating the prime and ultimate ratios of naſcent and evaneſcent finite quantities. And theſe are the principles of the calculation of Fluxions, viz. 1. Since Fluxions are in the prime ratio of their naſcent increments, and in the ultimate ratio of their evaneſcent (Art. 6.) 2. Fluxions may be expreſſed by thoſe increments when they firſt begin to be, or, laſtly, are vaniſhing. 3. Conſtant quantities have neither increments nor decrements, and therefore have no Fluxions. 4. If there be ſome quantities (among thoſe that are determinate) which decreaſe while others increaſe, the Fluxions of thoſe that decreaſe are negative; for they are as negative increments, or as decrements.

14. Flowing quantities are denoted by x, y, z, v, &c. the laſt letters of the alphabet, and conſtant quantities by a, b, c, d, &c.

The Fluxion of x is $\dot{x}$ with a dot over it, and Fluxion of $\dot{x}$ is $\ddot{x}$, or ſecond Fluxion.
of y $\dot{y}$ the Fluxion of $\dot{y}$ is $\ddot{y}$
of z $\dot{z}$ —— of $\dot{z}$ is $\ddot{z}$
of v $\dot{v}$ —— of $\dot{v}$ is $\ddot{v}$
&c. &c.

Sir *Iſaac Newton*, in his *Principia*, lays down this

LEMMA.

The moment of any *genitum* is equal to the moments of each of the generating ſides, drawn into the indices of the powers of thoſe ſides, and into their coefficients continually.

" He calls any quantity a *genitum*, which is not made by addition, or ſubduction of divers parts, but is generated or produced in arithmetic, by the multipli-

Fig. multiplication, diviſion, or extraction of roots of
3. any terms whatſoever; in geometry by the invention of contents and ſides, or of the extreams and mean proportionals." "Quantities of theſe ſorts, ſays he, I conſider as variable and indetermined, and increaſing or decreaſing, as it were, by a perpetual motion or flux; and I underſtand their momentaneous increments or decrements by the name of moments. But finite particles are not moments, but the very quantities generated by the moments; it will be the ſame thing, if inſtead of moments we uſe either the velocities of the increments and decrements (which may be called the *Motions*, *Mutations*, and *Fluxions* of quantities) or any finite quantities proportional to thoſe velocities."

Thus if the moments of any quantities A, B, C, &c. increaſe or decreaſe by a perpetual flux, or the velocities of the mutations which are proportional to them be called a, b, c, &c. ſo that while A becomes $A+a$, B becomes $B+b$, C becomes $C+c$ &c.

The moment of A^2 will be $= 2aA$
of A^3 —— $= 3aA^2$
of A^4 —— $= 4aA^3$
of $A^{\frac{1}{2}}$ —— $= \frac{1}{2}aA^{-\frac{1}{2}}$
of $A^{\frac{3}{2}}$ —— $= \frac{3}{2}aA^{\frac{1}{2}}$
of $A^{\frac{1}{3}}$ —— $= \frac{1}{3}aA^{-\frac{2}{3}}$
of $A^{\frac{2}{3}}$ —— $= \frac{2}{3}aA^{-\frac{1}{3}}$
of A^{-1} —— $= -aA^{-2}$
of A^{-2} —— $= -2aA^{-3}$
of $A^{-\frac{1}{2}}$ —— $= -\frac{1}{2}aA^{-\frac{3}{2}}$
&c. &c.

And in general that the moment of any power $A^{\frac{n}{m}}$ will be $= \frac{n}{m}aA^{\frac{n-m}{m}}$. And alſo the moment of A^2B will be $2aAB + bA^2$, the moment of the generated quantity $A^3B^4C^2$ will be $3aA^2B^4C^2 + 4bA^3B^3C^2$

$4bA^3B^3C^2 + 2cA^3B^4C$; and the moment of the generated quantity $\frac{A^3}{B^2}$ or A^3B^{-2}, will be $3aA^2B^{-2} - 2bA^3B^{-3}$ &c. and the rectangle AB will be $aB+bA$. Fig. 3.

15. Now if inſtead of A, B, C, &c. we uſe x, y, z, &c. and inſtead of the moments a, b, c, &c. we write $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. the ſenſe of the Lemma is, that the moment or Fluxion

of x^2 is $= 2x\dot{x}$
of x^3 —— $= 3x^2\dot{x}$
of x^4 —— $= 4x^3\dot{x}$
of $x^{\frac{1}{2}}$ —— $= \frac{1}{2}x^{-\frac{1}{2}}\dot{x}$
of $x^{\frac{3}{2}}$ —— $= \frac{3}{2}x^{\frac{1}{2}}\dot{x}$
of $x^{\frac{1}{3}}$ —— $= \frac{1}{3}x^{-\frac{2}{3}}\dot{x}$
of $x^{\frac{2}{3}}$ —— $= \frac{2}{3}x^{-\frac{1}{3}}\dot{x}$
of x^{-1} —— $= -x^{-2}\dot{x} = \frac{-\dot{x}}{x^2}$
of $\frac{1}{x^2}$ or x^{-2} $= -2x^{-3}\dot{x}$
of $x^{-\frac{1}{2}}$ —— $= -\frac{1}{2}x^{-\frac{3}{2}}\dot{x}$
of x^m —— $= mx^{m-1}\dot{x}$
of $x^{\frac{n}{m}}$ —— $= \frac{n}{m}x^{\frac{n-m}{m}}\dot{x}$, thus $x^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1-3}{3}}\dot{x} = \frac{1}{3}x^{-\frac{2}{3}}\dot{x}$.
of xy —— $= y\dot{x} + x\dot{y}$
of xyz —— $= yz\dot{x} + xz\dot{y} + xy\dot{z}$

15. The Fluxion of a quantity compoſed of ſeveral terms by + and —, is equal to all the Fluxions of all the terms joined together by the ſame ſigns + and —; thus the Fluxion of $a + x - y$ will be $\dot{x} - \dot{y}$.

DEMOMSTRATION.

The increment of the whole quantity $a + x - y$, generated in a given time, is equal to the difference

Fig. 3. of the increments of x and y; and ſince a is a conſtant quantity (Art. 14.) it hath no increment, therefore the naſcent or evaneſcent increment of $a + x - y$ is equal to the difference of the naſcent or evaneſcent increments of x and y; but Fluxions are in the prime ratio of their naſcent increments, therefore the Fluxion of the whole quantity $a + x - y$ is $\dot{x} - \dot{y}$. Q. E. D.

N. B. If while x increaſes y ſhould decreaſe, the Fluxion of y would be negative, namely, $-\dot{y}$, and therefore the Fluxion $\dot{x} - \dot{y}$ would become $\dot{x} + \dot{y}$, which always muſt be carefully obſerved.

16. Hence it is evident from the above Demonſtration, that Fluxions are not the velocities of the increments conſidered as actually generated, but as ariſing or beginning to be generated, and are exactly in the prime ratio of the increments, conſidered as in the firſt moment of their generation. “So that the genuine ſenſe and meaning, ſays Mr. *Hodgſon*, of finding the Fluxions of flowing or determinate quantities, is to find the velocities with which they ariſe, or begin to be generated in the firſt moments of formation. And the ſame way of reaſoning obtains, whether we conſider the firſt or laſt ratios of the velocities of the incremental parts; that is, whether we conſider them in their firſt ratios, as ariſing or beginning to be, or whether we conſider them in their laſt ratios, as vaniſhing and ceaſing to be.”

17. It has been ſaid, that the Fluxion of xy is $y\dot{x} + x\dot{y}$; as this is the main or fundamental point in the method of Fluxions, to find the Fluxion of the product of two flowing quantities, it will be neceſſary to give the following

DEMONSTRATION.

Let the right line BC move along the right line AB, to which it is perpendicular, and let C, the extreme

Fig. 4. extreme point thereof, deſcribe the curve ACc; let BC move to bc, and compleat the rectangles BF, bf; call AB, x, BC, y, conſequently the rectangle will be xy; while BC moves to bc, the increment of the rectangle BF or xy is equal to the ſum of the rectangles BE, EG, Cf; but the rectangle EG is to the rectangle EB, as Ec to BC, and to the rectangle Cf, as CE or Bb is to FC, or AB; bc returning again into its former place BC, and while Ec and EC decreaſe, and at laſt ultimately vaniſh, it decreaſes too, and at laſt vaniſhes; or the ratio of the rectangle EG to the rectangles EB and cf becomes unaſſignable, therefore (by Lem. 1. *Principia*) the ſum of the two rectangles BE, cf, become ultimately equal to the ſum of the three rectangles or parallelograms BE, EG, Cf; therefore the naſcent increment of the rectangle or parallelogram BF or xy is equal to the ſum of the two naſcent rectangles BE, Cf, or to the ſum of the products of x, into the naſcent increment of y, and of y into the naſcent increment of x; and therefore the Fluxion of the product of xy is $x\dot{y}+y\dot{x}$. Whence the Fluxion of ax is $a\dot{x}$, becauſe a is a conſtant quantity, and hath no Fluxion. Q. E. D.

Or thus.

The ſame may be proved another way from the abovesaid *Lemma*, *viz.*

DEMONSTRATION.

5. Let there be a rectangle OACB under two variable lines OA, OB, continually increaſing on each ſide A, take Ae, Af, equal, and Bg, Bh, equal parts on each ſide B, ſo that if a and b are quantities proportional to the moments of the lines OA, OB; ef be equal to a, and $gh = b$; compleat the rectangles $OgEe$, $OhFf$, and draw FE, which will paſs through C, the point of concourſe of the lines AC,

Fig. 5. AC, BC, (becauſe of the parallels, and the lines ef and gb, likewiſe biſected in A and B) I ſay, that the ſum of the trapeziums $EFef$ and $EFgb$ will be equal to the moment of the rectangle OACB; but the ſum of the trapeziums is had, by taking the difference of the rectangles $OeEG$ and $OfFH$, which is $Of \times Ob - Oe \times Og$; or it is $\overline{OA + Af} \times \overline{OB + Bb} - \overline{OA - Ae} \times \overline{OB - Bg}$; and putting $OA = A$, $OB = B$, $Af = Ae = \frac{1}{2}a$, $Bb = Bg = \frac{1}{2}b$, the difference of the rectangles will be $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b} - \overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b} = AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab - AB + \frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab = aB + bA$; and if we ſubſtitute x for A, y for B, and $\dot{x}$ for a, and $\dot{y}$ for b, it will be $x\dot{y} + y\dot{x}$, as before.

But to prove that the ſum of the trapeziums $EFef$ and $EFgb$ is equal to the moment of the rectangle OACB, if any lines, as ST, VX, howſoever unequal, be perpendiculars on SV, and TX be joined, and in the middle of SV there be erected 6. the perpendicular YZ, the trapezium STXV will be equal to the rectangle $SV \times YZ$; therefore the trapezium $EFef$ will be equal to the rectangle $AC \times ef$, and the trapezium $EFgb$ equal to the rectangle $BC \times gb$. Moreover, becauſe ef and gb are proportional to the moments of the lines OA, OB, that is, proportional to the velocities with which OA, OE, increaſe; or, which is the ſame thing, to the celerities by which, while the rectangle OACB increaſes, the lines AC, BC, are carried forward; the rectangles $AC \times ef$ and $BC \times gb$, will be as thoſe lines AC, BC, and their velocities conjunctly. But the Fluxion of the generated rectangle CACB is proportional to the cauſe which produces it, and that cauſe is the motion of the variable lines AC, BC, with which they are carried forward, while OA, OB, increaſe; and though while thoſe lines AC, BC, are carried forward, and the lines OA,

OB, may increaſe, we can get no ratio of this aug- Fig.
ment or increment, while we conſider the Fluxion, 6.
or the naſcent increment of the rectangle; for in the very beginning of this naſcent increment, theſe *productions* of OA, OB, are evidently nothing, and when firſt they are ſomething, others now, as AC, BC, are aſſumed greater than the former; therefore the momentum of the rectangle or parallelogram OAB, or the momentaneous cauſe of its mutation or Fluxion muſt be determined from AC and BC, and from the velocities with which they are carried on.

Let there be two parallelograms MN*mn*, PR*pr*, 7.
make MN = PR, and let others be conceived likewiſe equal to theſe which, flowing from MN and PR by an uniform and parallel motion, are carried forward in the direction of M*m* and P*p*, ſo that they may arrive at *mn* and *pr* in the ſame moment of time; it is manifeſt, that the areas M*n*, P*r*, will be as M*m*, P*p*, and likewiſe the velocities of thoſe lines flowing from MN and PR, will be in the ſame ratio, and conſequently the areas M*n*, P*r*, will be in the ratio of their velocities. But if MN, PR, be not equal, the areas will be as thoſe lines MN, PR, and their velocities conjunctly. And any increments of the rectangles MN*mn*, PR*pr*, made in
an equal time, will be in the ſame ratio; and there- 5.
fore the naſcent increments will be in that ratio too. Whence it follows, that the increment of the rectangle OABC, ariſing from the motion of AC, is as AC and its velocity conjunctly, and becauſe the increment of the ſame rectangle OABC, ariſing from the motion of BC, is as that line BC, and its velocity conjunctly; therefore the whole moment of the rectangle OACB is the ſum of the products of AC and BC, multiplied by the reſpective velocities with which they are carried forward, and conſe-

Fig. 5. quently as the ſum of the rectangles AC × *ef* and BC × *gh*, or finally as the ſum of the trapezia's EF*ef*, EF*gh*. Q. E. D.

II.

The following Eſſay on Fluxions I publiſhed in my Miſcellanea Curioſa Mathematica, *and as that book has been long out of print, I ſhall inſert part of it here, which perhaps may be more acceptable to ſome of my Readers than the above.*

A DEMONSTRATION *of the firſt principles of* FLUXIONS, *without uſing any infinitely ſmall Quantities, or vaneſcent Parallelograms.*

8. I. If two right angles AB and CD are ſuppoſed to be generated in the ſame time, by the uniform motion of two points A and C, then the velocities of the generating points will be to one another in the ſame proportion as the lines themſelves; and the proportion of theſe two lines will be directly as the velocities drawn into unity, or as any two ſpaces very ſmall, or of any length whatever, deſcribed in the ſame time by the generating points. Therefore as the method of ſubſtituting algebraical characters for lines is *arbitrary*, as for inſtance, we may call AB, *y*, and CD, *x*, ſo we may expreſs the velocities of the points, whereby theſe two lines are uniformly generated, by what characters we pleaſe, for *example*, by $\dot{y}$ and $\dot{x}$, the former letters with a point over them; or theſe velocities may be repreſented by two lines A*a* and C*c*, paſſed over in the ſame time by the points A and C. The velocities of the points A and C are the *Fluxions* of the lines AB and CD, which are properly expreſſed by $\dot{y}$ and $\dot{x}$, or $1\dot{y}$ and $1\dot{x}$, and repreſented by A*a* and C*c*, any two lines bearing the ſame proportion to one another that the velocities do.

Again, the line AB may be conceived to be traced Fig. out gradually by a point moving on from A to B, 8. either with an equable motion, or with a velocity in any manner varied. And the velocity or degree of ſwiftneſs, with which this point moves in any part of the line AB, is called the *Fluxion* of this line at that place.

Fluxions then in general are the velocities with which magnitudes, varying by a continued motion, increaſe or diminiſh; and the magnitudes themſelves are reciprocally called the *Fluents* of thoſe *Fluxions*.

And as different *Fluents* may be underſtood to be deſcribed together in ſuch manner, as conſtantly to preſerve ſome one known relation to each other, the doctrine of Fluxions teaches how to aſſign, at all times, the proportion between the velocities wherewith homogeneous magnitudes, varying thus together, augment or diminiſh.

This doctrine alſo teaches, on the other hand, how, from the relation known between the Fluxions, to diſcover what relation the Fluents themſelves bear to each other. It is by means of this proportion only, that *Fluxions* are applied to geometrical uſes; for this doctrine never requires any degree of velocity to be aſſigned for the Fluxion of any one Fluent; and the proportion between the Fluxions of magnitudes is aſſignable from the relation known between the magnitudes themſelves.

II. If the two rectangles AD and EH are ſup- 9. poſed to be generated in the ſame time by the uni- 10. form motion of the lines AB and EF, always parallel to themſelves, along the lines BD and FH, then the rectangle AD will be to the rectangle EH, as AB drawn into the velocity of its motion is to EF, drawn into the velocity of its motion; or AB into the velocity of the point B is to EF into the

velocity

Fig. velocity of the point F, as the rectangle AD to the
10. rectangle EH.

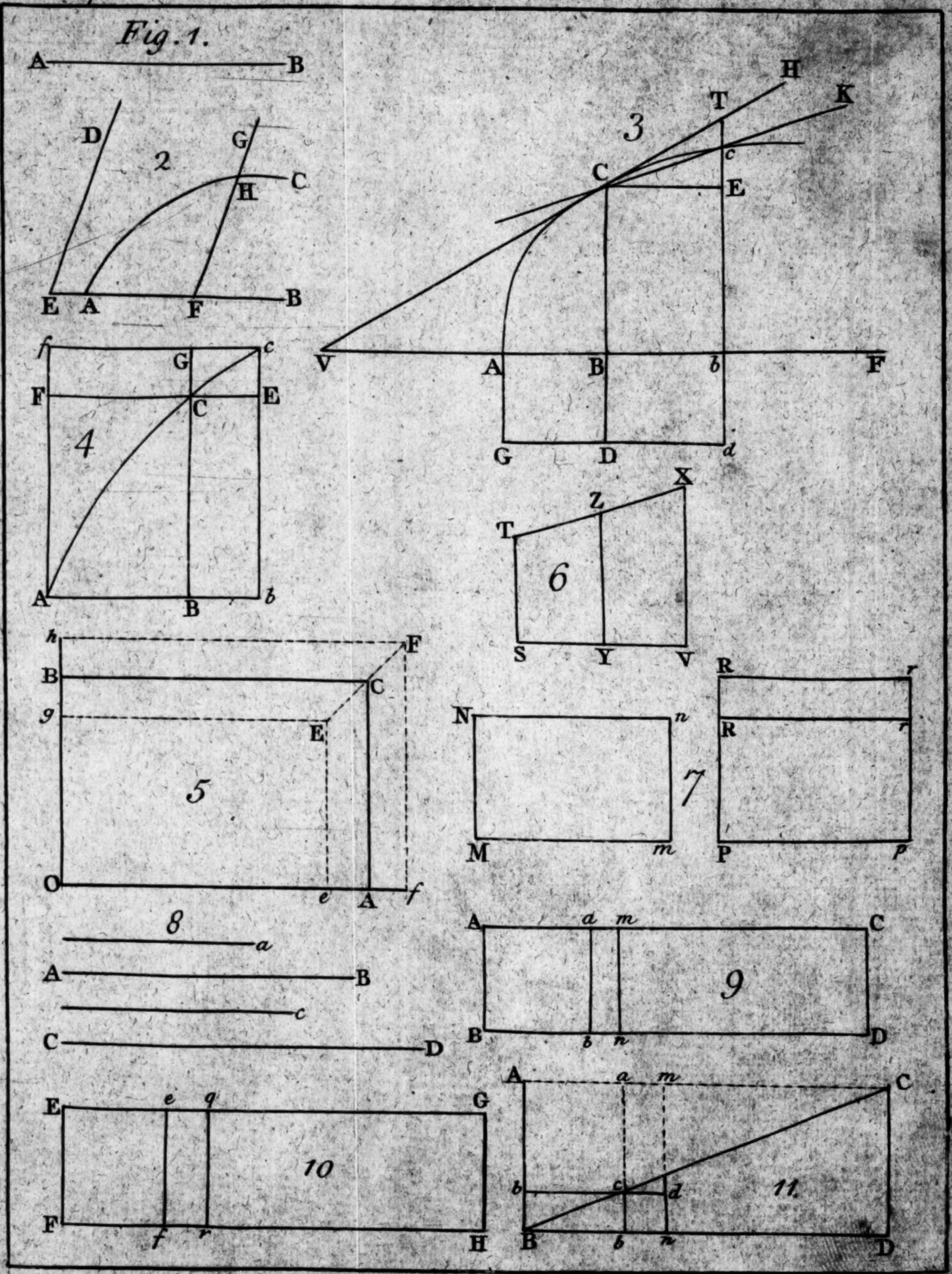
That is, if we call AB, a; EF, b; BD, y; FH, x; and the Fluxions of BD, $\dot{y}$, of FH, $\dot{x}$, as before; or repreſent them by the ſpaces B*b* and F*f*, or *bn* and *fr*, paſſed over in the ſame time by the lines AB and EF; then will $a\dot{y}$ be to $b\dot{x}$, as the rectangle AD to the rectangle EH: and the velocity wherewith the line EF is moved, drawn into that line, is the *Fluxion* of the rectangle EH; which are properly expreſſed by $a\dot{y}$ and $b\dot{x}$, and may be repreſented by the rectangles A*b* and E*f*, or *an* and *er*, deſcribed in the ſame time by the uniform motion of the lines AB and EF.

11. III. But if the two lines AB and EF ſhould be-
12. come variable, and flow uniformly along BD and FH, the lines DC and GH ſhould increaſe uniformly from *o*, or the points B and F, till they become DC and HG; then it is evident, that by ſuch a motion, inſtead of the rectangles AD and EH, there will only be deſcribed the triangles BDC and FHG. And the triangle BDC will be to the triangle FHG, as the rectangle of the generating lines (taken at the ſame time) into the velocities wherewith they are moved along; that is, ſuppoſing DC $= u$, and HG $= z$, at any time the Fluxions of the triangles may be expreſſed by $u\dot{y}$ and $z\dot{x}$ (ſuppoſing BD and FH repreſented as above) and may be repreſented by the rectangles *bb* and *ff*, or *cn* and *gr*, deſcribed in the ſame time by the lines *bc* and *fg*, taken at the length they are in any one moment of time. Hence in general,

IV. The Fluxion of any quantity whatever is equal to the rectangle or product of the generating quantity, drawn into the velocity of its motion; and may be repreſented by the rectangle of the generating quantity, at any point of time, drawn into

a line

Pl. I. *pa. 88.*



Fluxions

a line proportional to the velocity. This definition Fig.
is plain and general, being eaſily underſtood, and 12.
applicable to all Fluxions; as I ſhall afterwards make appear.

V. If the Fluxion of the rectangle of two variable quantities be required, or by what motion of two flowing quantities the rectangled parallelogram EFHG may be deſcribed, I anſwer thus;

Let the point F move uniformly along the line 13.
FH, while at the ſame time the line HG uniformly increaſes from *o* at the point F, till it becomes equal to GH; then it is evident, that by that motion there will be deſcribed the triangle FHG. And at the ſame time ſuppoſe a point from F moves uniformly to D, while the line DG increaſes uniformly from *o* at the point F, till it become = DG; then, for the ſame reaſon, by that motion will be deſcribed the triangle FDG. But theſe two compleat the rectangles; therefore if we ſuppoſe FH = DG = x, and FD = GH = y, and expreſs their Fluxions by $\dot{x}$ and by $\dot{y}$, or repreſent them by the ſpaces *fb* and *dc*, paſſed over in the ſame time, the Fluxion of the triangle FHG will be properly expreſſed by $y\dot{x}$, and repreſented by the rectangle *bb*; and the Fluxion of the triangle FDG expreſſed by $x\dot{y}$, and repreſented by *cb*, by Art. III. Hence the Fluxion of the whole rectangle DFHG is properly expreſſed by $y\dot{x} + x\dot{y}$, and repreſented by the two rectangles *cb* and *bb*; I ſay, *repreſented* by theſe two rectangles, which are proportional to the Fluxions, but are not the Fluxions themſelves. For the Fluxions are rectangles under the generating quantities and velocities, and ſo far as velocity, which is a property of motion, may be repreſented by a right line, ſo far and no farther, may the Fluxion of the rectangle be repreſented by two rectangles *cb* and *bb*, provided the lines *cd* and *fb* bear the ſame propor-

tion

Fig. portion to one another that the velocities do, wherewith the lines FD and FH are generated.

14. VI. Hence it will be eaſy to find the Fluxion of the parallelopipedon AH; for put AB $= a$, AD $= b$, AF $= x$, and its Fluxion $= \dot{x}$, the Fluxion of the ſolid will be $ab\dot{x}$, and may be repreſented by the little ſolid ABCD *abcd*, being ſuppoſed to be generated by the uniform motion of the plane ABCD along the line AF with an uniform velocity.

15. VII. In like manner if the lines AB $= x$, AD $= y$, and AF $= z$, be all variable, and their Fluxions be $\dot{x}$, $\dot{y}$, and $\dot{z}$, then while x and y move uniformly along AF, they will deſcribe the pyramid AFGHE, and the Fluxion will be $xy\dot{z}$. Suppoſing them to increaſe from o till they become FE and FG; z and y by an uniform motion from A to BE, and BC *by*, and uniformly increaſing, will deſcribe the pyramid AEHCB, and their Fluxion will be $zy\dot{x}$. And, laſtly, while z and x move uniformly from the point A along the line AD, and uniformly increaſe till they become DG and DC, they will by ſuch a motion deſcribe the pyramid ACDGH, the Fluxion of which will be $zx\dot{y}$. But theſe three pyramids compoſe the parallelopipedon, and therefore the ſum of their Fluxions, viz. $zx\dot{y} + zy\dot{x} + xy\dot{z}$ will be the Fluxion of that ſolid. Q. E. I.

Hence the Fluxion of xy is $x\dot{y} + y\dot{x}$; and the Fluxion of xyz is $xy\dot{z} + xz\dot{y} + zy\dot{x}$ $(= xy\dot{z} + x\dot{y}z + \dot{x}yz)$; but if $x = y = z$, then the Fluxion of xx or x^2 is $= x\dot{x} + \dot{x}x = 2x\dot{x}$, and the Fluxion of xxx or x^3 is $= xx\dot{x} + x\dot{x}x + \dot{x}xx = 3xx\dot{x} = 3x^2\dot{x}$. Mr. *Emerſon* has proved this from other principles, at page 23 of his method of *Increments*, which ſee.

16. VIII. But now let us ſuppoſe, that the line HG ſhould not increaſe uniformly, but that its ſquare ſhould always bear a certain proportion to the ſpace paſſed over by the point F, then the line FG, in-

ſtead of being a right one, will become a curve, and the area FGH not equal to the area of the triangle in Art. III. Yet ſtill ſuppoſing that ſpace generated, in the ſame time, by the uniform motion of the line HG, from the point F to GH, the Fluxion may be properly expreſſed by $y\dot{x}$ (calling GH, y, FH, x; and its Fluxion $\dot{x}$) and repreſented by the rectangle *ff* or *gr*, the difference here, and in Art. III, conſiſting only in y, or the line *gf*, or the proportion that GH bears to FH. Hence alſo it will be eaſy to find the Fluxion of a ſolid, generated by the rotation of any curve around its axis, or by the uniform motion of a circle along a right line, whoſe radius increaſes in any given proportion. Fig. 16.

THE

THE
DIRECT METHOD OF FLUXIONS.

PROP. I.

To find the Fluxion of any ſimple variable quantity.

RULE.

PUT a dot over the variable quantity, and it is the Fluxion required.

EXAMPLES.

1ſt, The Fluxion of x is $\dot{x}$. 2d, The Fluxion of y is $\dot{y}$. 3d, The Fluxion of z is $\dot{z}$. 4th, The Fluxion of $x + y$ is $\dot{x} + \dot{y}$. 5th, The Fluxion of $a + x$ is $\dot{x}$. 6th, The Fluxion of $b + y$ is $\dot{y}$, for a and b are conſtant quantities, and have no Fluxions.

Again, 1ſt, The Fluxion of $x - y$ is $\dot{x} - \dot{y}$. 2d, Of $a - y$ is $-\dot{y}$. 3d, Of $a - x$ is $-\dot{x}$. 4th, The Fluxion of $x + y + z - v$ is $\dot{x} + \dot{y} + \dot{z} - v$. 5th, Of $x - z - y$ is $\dot{x} - \dot{z} - \dot{y}$. 6th, The Fluxion of $b + x - c + z + a - y$ will be $\dot{x} + \dot{z} - \dot{y}$; for b, c, a, are conſtant quantities, never increaſing or decreaſing, and cannot undergo any change, and therefore (by Art. XIII.) have no Fluxion.

PROP. II.

To find the Fluxion of any given power of any variable quantity.

RULE.

Multiply the index of the given power by the Fluxion of the root, and then multiply that product

duct by the power of the root whoſe index is leſſened by 1, and the coefficient of the term together, gives the Fluxion of that term required.

This may more briefly be expreſſed in algebraical characters, thus $max^{m-1}\dot{x}$ is the Fluxion of ax^m, by the above Rule.

EXAMPLES.

1. The Fluxion of x^2 is $= 2x^{2-1}\dot{x} = 2x\dot{x}$
2. —— of x^3 is $= 3x^{3-1}\dot{x} = 3x^2\dot{x}$
3. —— of x^4 is $= 4x^{4-1}\dot{x} = 4x^3\dot{x}$
4. —— of $7ax^5$ is $= 5 \times 7ax^{5-1}\dot{x} = 35ax^4\dot{x}$.

PROP. III.

To find the Fluxion of the product of ſeveral variable quantities.

RULE.

Multiply the Fluxion of each one by the product of the reſt of the quantities, and the ſum of the products thus ariſing will be the Fluxion ſought, Art. XVII.

Or,

The ſum of the products of the Fluxion of each factor, multiplied by the product of the other factors, is the Fluxion of the quantities when multiplied together.

The ſame may be expreſſed in algebraical characters thus, univerſally,

I. The Fluxion of $x^m y^n$ is $= y^n m x^{m-1}\dot{x} + x^m n y^{n-1}\dot{y}$.

II. The Fluxion of $x^m y^n z^c$ is $= y^n z^c m x^{m-1}\dot{x} + x^m z^c n y^{n-1}\dot{y} + x^m y^n c z^{c-1}\dot{z}$.

III. The Fluxion of $x^m y^n z^c v^p$ is $= y^n z^c v^p m x^{m-1}\dot{x} + x^m z^c v^p n y^{n-1}\dot{y} + x^m y^n v^p c z^{c-1}\dot{z} + x^m y^n z^c p v^{p-1}\dot{v}$.

&c.

EXAM.

EXAMPLES.

1. The Fluxion of xy is $= y\dot{x} + x\dot{y}$, here m and n are each $= 1$.
2. —— of zv is $= v\dot{z} + zv$, here m and n are each $= 1$.
3. —— of axy is $= ax\dot{y} + ay\dot{x}$.
4. —— of xyz is $= yz\dot{x} + xz\dot{y} + xy\dot{z}$.

Or thus.

Put $zy = v$, then xyz will be $= vx$, and conſequently the Fluxion of xyz is $=$ Fluxion of vx; but the Fluxion of vx is $x\dot{v} + v\dot{x}$; and $\dot{v} = z\dot{y} + y\dot{z}$; now if in the Fluxion of $x\dot{v} + v\dot{x}$ there be wrote zy for v, and $z\dot{y} + y\dot{z}$ for $\dot{v}$, we ſhall have $xz\dot{y} + yx\dot{z} + zy\dot{x} = yz\dot{x} + xz\dot{y} + xy\dot{z}$, and the like way of reaſoning of other products.

5. The Fluxion of $xyzv$ will be $yzv\dot{x} + xzv\dot{y} + xyv\dot{z} + xyzv$.

6. The Fluxion of $x^2y^3z^4v^5$ will be $= y^3z^4v^5\,2x^{2-1}\dot{x} + x^2z^4v^5\,3y^{3-1}\dot{y} + x^2y^3v^5\,4z^{4-1}\dot{z} + x^2y^3z^4\,5v^{5-1}\dot{v} = 2y^3z^4v^5x\dot{x} + 3x^2z^4v^5y^2\dot{y} + 4x^2y^3v^5z^3\dot{z} + 5x^2y^3z^4v^4\dot{v}$.

PROP. IV.

Given any equation involving Fluents or flowing quantities to find the Fluxion.

RULE.

Multiply every term of the equation ſeparately by the indices of the powers of the Fluents that are in it; and in every ſuch product, change one of the roots of the powers into its Fluxion, the ſum of all the products joined with their proper ſigns, will be the Fluxion of the equation ſought.

EXAM.

EXAM. 1.

What is the Fluxion of this equation $x^4 - xy^3 + a^2z^2 + bv = d^2x$.

1. The Fluxion of x^4 will be $4x^3\dot{x}$ by *Prop.* II.
2. —— of $-xy^3$ will be $-\dot{x}y^3 - 3xy^2\dot{y}$
3. —— of $+a^2z^2$ will be $2a^2z\dot{z}$.
4. —— of $+bv$ will be $b\dot{v}$.
5. —— of d^2x will be $d^2\dot{x}$.

Therefore $4x^3\dot{x} - \dot{x}y^3 - 3xy^2\dot{y} + 2a^2z\dot{z} + b\dot{v} = d^2\dot{x}$ is the Fluxion required.

EXAM. 2.

What is the Fluxion of $ax^3 - bx^2 + cxy - v - y^5 = 0$.

1. The Fluxion of ax^3 is $= 3ax^2\dot{x}$.
2. —— of $-bx^2$ is $= -2bx\dot{x}$.
3. —— of $+cxy$ is $= cy\dot{x} + cx\dot{y}$.
4. —— of $-v$ is $= -\dot{v}$.
5. —— of $-y^5$ is $= -5y^4\dot{y}$.

Therefore the Fluxion of $ax^3 - bx^2 + cxy - v - y^5$, will be $= 3ax^2\dot{x} - 2bx\dot{x} + cy\dot{x} + cx\dot{y} - \dot{v} - 5y^4\dot{y} = 0$.

EXAM. 3.

Let $4x^4 + yx^2 - 9bzy = z^4 - 5yz^3$ *be put into Fluxions.*

The Fluxion is $= 16x^3\dot{x} + 2yx\dot{x} + x^2\dot{y} - 9bz\dot{y} - 9by\dot{z} = 4z^3\dot{z} - 15yz^2z - 5z^3\dot{y}$.

PROP. V.

To find the Fluxion of any fraction.

RULE.

Multiply the Fluxion of the numerator by the denominator, then multiply the Fluxion of the denominator into the numerator, ſubtract this laſt product from the former, and divide the remainder by the ſquare of the denominator, gives the Fluxion required.

Or univerſally.

The Fluxion of $\frac{x^m}{y^n}$ is $= \frac{y^n m x^{m-1}\dot{x} - x^m n y^{n-1}\dot{y}}{y^{2n}}$ $= \frac{\dot{N}D - N\dot{D}}{DD}$. Where N denotes the numerator and D the denominator.

EXAM. I.

What is the Fluxion of $\frac{x}{y}$, *or* xy^{-1}?

1ſt, $\dot{x} \times y = y\dot{x}$; and, 2d, $\dot{y} \times x = x\dot{y}$: then $\frac{y\dot{x} - x\dot{y}}{yy}$ the Fluxion. For $\frac{x}{y}$ or $x \times \frac{1}{y}$ is $= \dot{x} \times \frac{1}{y} - \frac{\dot{y}}{yy} \times x = \frac{y\dot{x} - x\dot{y}}{yy}$.

Or thus.

Put $\frac{x}{y} = z$, then $x = yz$, in Fluxions $\dot{x} = y\dot{z} + z\dot{y}$, but $y\dot{z} = \dot{x} - z\dot{y}$, and $\dot{z} = \frac{\dot{x}}{y} - \frac{z\dot{y}}{y}$, for z in the expreſſion $\frac{-z\dot{y}}{y}$ put its equal $\frac{x}{y}$; then $\frac{\dot{x}}{y} - \frac{x\dot{y}}{yy} = \frac{yy\dot{x} - xy\dot{y}}{y^3} = \frac{y\dot{x} - x\dot{y}}{yy}$.

EXAM. 2.

Required the Fluxion of $\frac{z}{y}$.

Then the Fluxion of $\frac{z}{y}$ is $= \frac{y\dot{z} - z\dot{y}}{yy}$ by the ſame way of proceeding.

EXAM. 3.

Let $\frac{a}{x}$ *or* ax^{-1} *be put into a Fluxion.*

Then ($a \times \frac{1}{\dot x}$ is $= -\frac{\dot x}{xx} \times a) = -\frac{a\dot x}{xx}$.

Alſo the Fluxion of $\frac{a}{x^3}$ will be $= -\frac{3a\dot x}{x^4}$.

Univerſally.

The Fluxion of $\frac{a}{x^m}$ will be $= -\frac{ma\dot x}{x^{m+1}}$.

EXAM. 4.

What's the Fluxion of $\frac{z}{a+z}$ *?*

1. $\dot z \times \overline{a+z} = a\dot z + z\dot z$.

2. $z \times \dot z = z\dot z$. Therefore $\frac{a\dot z + z\dot z - z\dot z}{a^2 + 2az + zz} = \frac{a\dot z}{\overline{a+z}|^2}$.

EXAM. 5.

Put $\frac{1}{\dot x}$ *into a Fluxion.*

Thus the Fluxion of $\frac{1}{x}$ or x^{-1} is $= \frac{-1 \times \dot x}{xx} = \frac{-\dot x}{xx}$.

Alſo the Fluxion of $\frac{1}{x^3}$ or x^{-3} is $= \frac{-3 \times \dot x}{x^4} = -\frac{3\dot x}{x^4}$.

Univerſally.

The Fluxion of $\frac{1}{x^m}$ or x^{-m}, will be $-m\dot x x^{-m-1} = -m\dot x \times \frac{1}{x^{m+1}} = -\frac{m\dot x}{x^{m+1}}$.

EXAM. 6.

Put $\frac{x}{x+y}$ *into Fluxions.*

1. $\dot{x}\times\overline{x+y} = x\dot{x}+y\dot{x} = \dot{N}\times D$.
2. $\overline{\dot{x}+\dot{y}}\times x = x\dot{x}-x\dot{y} = \dot{D}\times N$.
3. Then $\overline{x+y}|^2 = x^2+2xy+yy = DD$.

Therefore $\frac{x\dot{x}+y\dot{x}-x\dot{x}-x\dot{y}}{x^2+2xy+yy} = \frac{y\dot{x}-x\dot{y}}{a^2+2xy+yy}$, the Fluxion ſought, $=\frac{\dot{N}D-N\dot{D}}{DD}$.

EXAM. 7.

Put $\frac{x+y+z}{x+y}$ *into Fluxions.*

1. $\frac{x+y+z}{x+y} = 1+\frac{z}{x+y}$.

Then 2. $\dot{z}\times\overline{x+y} = \dot{z}x+\dot{z}y = \dot{N}\times D$.

3. $\overline{x+\dot{y}}\times z = z\dot{x}+z\dot{y} = \dot{D}\times N$.

Therefore $\frac{x\dot{z}+y\dot{z}-z\dot{x}+z\dot{y}}{x^2+2xy+yy}$, the Fluxion required, $=\frac{\dot{N}D-N\dot{D}}{DD}$.

EXAM. 8.

Required the Fluxion of $\frac{x^2}{y^2}$.

1. $2x\dot{x}\times y^2 = 2y^2x\dot{x}$.
2. $2y\dot{y}\times x^2 = 2x^2y\dot{y}$.

Therefore $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$ is the Fluxion required.

EXAM. 9.

To put $\frac{1}{x^{\frac{1}{3}}}$ *into Fluxions.*

The Fluxion is $= -\frac{1}{3}x^{-\frac{1}{3}-1}\dot{x} = -\frac{1}{3}x^{-\frac{4}{3}}\dot{x} = -\frac{1}{3}\dot{x} \times x^{\frac{1}{\frac{4}{3}}} = -\frac{\frac{1}{3}\dot{x}}{x^{\frac{4}{3}}}$.

EXAM. 10.

To put $\frac{x^3}{y^5}$ *into Fluxions.*

1. $3x^2\dot{x} \times y^5 = 3y^5x^2\dot{x}$.
2. $5y^4\dot{y} \times x^3 = 5x^3y^4\dot{y}$.

Therefore $\frac{3y^5x^2\dot{x} - 5x^3y^4\dot{y}}{y^{10}}$ is the Fluxion required.

EXAM. 11.

To put $\frac{x^2z^2}{y}$ *into Fluxions.*

$$\left.\begin{array}{l}\text{1. The Fluxion of } z^2 \text{ in } x^2z^2 \text{ will be } = 2x^2z\dot{z}\\ \text{2. The Fluxion of } x^2 \text{ in } x^2z^2 \text{ will be } = 2z^2x\dot{x}\end{array}\right\}\dot{N}.$$

3. Then $\overline{2x^2z\dot{z} + 2z^2x\dot{x}} \times y = 2yx^2z\dot{z} + 2yz^2x\dot{x} = \dot{N}D$.

4. And $x^2z^2 \times \dot{y} = x^2z^2\dot{y} = \dot{D}N$.

5. Therefore $\frac{2yx^2z\dot{z} + 2yz^2x\dot{x} - x^2z^2\dot{y}}{yy}$ is the Fluxion required.

Univerſally.

The Fluxion of $\frac{1}{x^{m+1}z^{n+1}}$ will be $= -\frac{\overline{n+1}.x\dot{z} + \overline{m+1}.z\dot{x}}{x^{m+2}z^{n+2}}$. See Mr. *Emerſon*'s method of Increments, page 23.

PROP. VI.

To find the Fluxion of powers where ſurds are concerned.

RULE.

RULE.

This is the ſame Rule as in *Prop.* II. viz. Multiply the index, the Fluxion of the root, its power, whoſe index is leſſened by 1, and the coefficient of the term continually, for the Fluxion of that term.

This may more briefly be expreſſed in algebraical characters as a general *formula*, thus.

I. $x^{\frac{1}{n}}$ will be in Fluxions $= \frac{1}{n} x^{\frac{1}{n}-1} \dot{x}$.

II. The Fluxion of $x^{\frac{m}{n}}$ is $= \frac{m}{n} x^{\frac{m}{n}-1} \dot{x} = \frac{m}{n} x^{\frac{m-n}{n}} \dot{x}$.

III. The Fluxion of $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is $= -\frac{m}{n} x^{-\frac{m}{n}-1} \dot{x}$.

EXAM. 1.

Required the Fluxion of $\sqrt{x} = x^{\frac{1}{2}}$?

1. $\frac{1}{2}x x^{\frac{1}{2}-1} = \frac{1}{2}\dot{x} \times x^{-\frac{1}{2}} = \frac{1}{2} \times \frac{\dot{x}}{\sqrt{x}} = \frac{\dot{x}}{2\sqrt{x}}$.

Or by *Formula* I.

Here $n = 2$, then $\frac{1}{2} x^{\frac{1}{2}-1}\dot{x} = \frac{1}{2} x^{-\frac{1}{2}}\dot{x} = \frac{\dot{x}}{2\sqrt{x}}$.

EXAM. 2.

Required the Fluxion of $\sqrt[3]{x} = x^{\frac{1}{3}}$?

Here $n = 3$; then $\frac{1}{3} x^{\frac{1}{3}-1}\dot{x} = \frac{1}{3} x^{\frac{1-3}{3}} \dot{x} = \frac{1}{3} x^{-\frac{2}{3}}\dot{x} = \frac{\dot{x}}{3 \times x^{\frac{2}{3}}}$.

EXAM. 3.

What's the Fluxion of $\sqrt[3]{xx} = \overline{xx}|^{\frac{2}{3}}$?

By *Formula* II.

Here $m = 2$, $n = 3$; then $\frac{m}{n} x^{\frac{m}{n} - 1} \dot{x} = \frac{2}{3} x^{\frac{2}{3} - 1} \dot{x} = \frac{2}{3} x^{-\frac{1}{3}} \dot{x} = \frac{2x\dot{x}}{3x^{\frac{1}{3}}}$.

COROLLARIES.

1. Hence the Fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$, is $= \frac{3}{2} x^{\frac{3}{2} - 1} \dot{x} = \frac{3}{2} x^{\frac{1}{2}} \dot{x}$.

2. —— of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$, is $= \frac{3}{4} x^{\frac{3}{4} - 1} \dot{x} = \frac{3}{4} x^{-\frac{1}{4}} \dot{x}$.

3. —— of $\sqrt[4]{x^5}$ or $x^{\frac{5}{4}}$, is $= \frac{5}{4} x^{\frac{5}{4} - 1} \dot{x} = \frac{5}{4} x^{\frac{1}{4}} \dot{x}$.

EXAM. 4.

Put $\sqrt{aa + xx} = \overline{aa + xx}|^{\frac{1}{2}}$ *into Fluxions.*

1. $\frac{1}{2} \times \overline{aa + xx}^{\frac{1}{2} - 1} = \times + 2x\dot{x} = \frac{1}{2} \times \overline{aa + xx}^{-\frac{1}{2}} \times 2x\dot{x} = \frac{2x\dot{x}}{2\sqrt{aa + xx}} = \frac{x\dot{x}}{\sqrt{aa + xx}}$.

Or thus.

Put $z = \sqrt{aa + xx}$; then $zz = aa + xx$, in Fluxions $2z\dot{z} = 2x\dot{x}$; and by diviſion $\dot{z} = \frac{2x\dot{x}}{2z} = \frac{2x\dot{x}}{2\sqrt{aa + xx}} = \frac{x\dot{x}}{\sqrt{aa + xx}}$ the Fluxion required.

EXAM. 5.

Given $\sqrt{aa - xx} = \overline{aa - xx}^{\frac{1}{2}}$ *to be put into Fluxions.*

Then $\frac{1}{2} \times \overline{aa - xx}^{\frac{1}{2} - 1} \times - 2x\dot{x} = \frac{1}{2} \times \overline{aa - xx}^{-\frac{1}{2}} \times - 2x\dot{x} = \frac{-2x\dot{x}}{2\sqrt{aa - xx}} = \frac{-xx}{\sqrt{aa - xx}}$.

Or thus.

Put $z = \sqrt{aa - xx}$, then $z^2 = aa - xx$, but $2z\dot{z} = - 2x\dot{x}$, then $\dot{z} = \frac{-2x\dot{x}}{2z} = \frac{-2x\dot{x}}{2\sqrt{aa - xx}} = \frac{-x\dot{x}}{\sqrt{aa - xx}}$.

EXAM. 6.

Find the Fluxion of $\sqrt{2rx - xx} = \overline{2rx - xx}|^{\frac{1}{2}}$.

Then $\frac{1}{2} \times \overline{2rx - xx}|^{\frac{1}{2}-1} \times 2r\dot{x} - 2x\dot{x} = \frac{2r\dot{x} - 2x\dot{x}}{2\sqrt{2rx - xx}} = \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$.

Or thus.

Put $z = \sqrt{2rx - xx}$, then $z^2 = 2rx - xx$, in Fluxions $2z\dot{z} = 2r\dot{x} - 2x\dot{x}$, and $\dot{z} = \frac{2r\dot{x} - 2x\dot{x}}{2z} = \frac{2r\dot{x} - 2x\dot{x}}{2\sqrt{2rx - xx}} = \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$ the Fluxion ſought.

EXAM. 7.

To put $\overline{ax - yy}|^{3}$ *into Fluxions.*

Then $\overline{ax - yy}|^{3} = \overline{ax - yy}|^{\frac{3}{1}}$ is $= 3 \times \overline{ax - yy}|^{2} \times \overline{a\dot{x} - 2y\dot{y}} = \overline{9a^2x^2 - 18axy^2 + 9y^4} \times a\dot{x} - 2y\dot{y} = 9a^3x^2\dot{x} - 18a^2y^2x\dot{x} + 9ay^4\dot{x} - 18a^2x^2y\dot{y} + 36axy^3\dot{y} - 18y^5\dot{y} = 3 \times$ into $3a^3x^2\dot{x} - 6a^2y^2x\dot{x} - 3ay^4\dot{x} - 6a^2x^2y\dot{y} + 12axy^3\dot{y} - 6y^5\dot{y}$, the Fluxion required.

EXAM. 8.

Let it be required to find the Fluxion of $x\sqrt{aa + xx} = x \times \overline{aa + xx}|^{\frac{1}{2}}$ *?*

Put $x\sqrt{aa + xx} = z$, then $zz = x^2 \times \overline{aa + xx}$ or $zz = aax^2 + x^4$, in Fluxions $2z\dot{z} = 2a^2x\dot{x} + 4x^3\dot{x}$, and $\dot{z} = \frac{2a^2x\dot{x} + 4x^3\dot{x}}{2z} = \frac{2a^2x\dot{x} + 4x^3\dot{x}}{2x\sqrt{aa + xx}} = \frac{a^2\dot{x} + 2x^2\dot{x}}{\sqrt{aa + xx}}$ the Fluxion required.

Or thus.

$x\sqrt{aa + xx}$ may be put into an infinite ſeries, and its Fluxion taken in a ſeries.

I. ✓

1. $\sqrt{aa+xx} = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. by *Ex.* 2, of extraction of roots; and multiplying the ſeries by $\dot{x}$, we ſhall have $a\dot{x} + \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} + \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ &c. the Fluxion of $x\sqrt{aa+xx}$.

Cor.

And by the ſame way of proceeding the Fluxion of $x\sqrt{aa-xx} = \frac{a^2\dot{x}-2x^2\dot{x}}{\sqrt{aa-xx}}$.

Or in a ſeries,

$$a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7} \text{ \&c.}$$

Exam. 9.

Put $xx\sqrt{ay+xx} = xx \times \overline{ay+xx}|^{\frac{1}{2}}$ *into Fluxions.*

Let $z = xx\sqrt{ay+xx}$, then $zz = x^4 \times \overline{ay+xx}$, or $zz = ayx^4 + x^6$; in Fluxions $2z\dot{z} = 4ayx^3\dot{x} + 6x^5\dot{x} + ax^4\dot{y}$, and $\dot{z} = \frac{4ayx^3\dot{x} + 6x^5\dot{x} + ax^4\dot{y}}{2z} = \frac{4ayx^3\dot{x}+6x^5\dot{x}+ax^4\dot{y}}{2xx\sqrt{ay+xx}} = \frac{4ayx\dot{x}+6x^3\dot{x}+ax^2\dot{y}}{2\sqrt{ay+xx}}$.

Exam. 10.

Put $\sqrt{ax+\sqrt{aa+xx}}$ *into Fluxions.*

1. The Fluxion of $\sqrt{aa+xx} = \frac{x\dot{x}}{\sqrt{aa+xx}}$ by *Ex.* 4.

2. Put $z = \sqrt{ax+\sqrt{aa+xx}}$; then $zz = ax + \sqrt{aa+xx}$, in Fluxions.

3. $2z\dot{z} = a\dot{x} + \frac{x\dot{x}}{\sqrt{aa+xx}}$; but $\dot{z} = \frac{a\dot{x} + \frac{x\dot{x}}{\sqrt{aa+xx}}}{2z}$, therefore $\frac{ax + \frac{x\dot{x}}{\sqrt{aa+xx}}}{2\sqrt{ax+\sqrt{aa+xx}}}$ is the Fluxion required.

EXAM. 11.

To find the Fluxion of $\sqrt{a^2+x^2}|^3 = \overline{a^2+x^2}|^{\frac{3}{2}}$.

Then $\frac{3}{2} \times \overline{a^2+x^2}|^{\frac{3}{2}-1} \times 2x\dot{x} = \frac{3}{2} \times \overline{a^2+x^2}|^{\frac{1}{2}} \times 2x\dot{x} = \frac{6x\dot{x}}{2}\sqrt{a^2+x^2} = 3x\dot{x}\sqrt{a^2+x^2}$ the Fluxion required.

Or by infinite ſeries.

Then $3x\dot{x}\sqrt{a^2+x^2} = 3x\dot{x} \times : a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ or $3ax\dot{x} + \frac{3x^3\dot{x}}{2a} - \frac{3x^5\dot{x}}{8a^3} + \frac{3x^7\dot{x}}{16a^5} - \frac{3\cdot5x^9\dot{x}}{128a^7}$ &c.

EXAM. 12.

To find the Fluxion of $\sqrt{xy} = \overline{xy}|^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$.

Then $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^{\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\dot{y}x^{\frac{1}{2}} = \frac{\dot{x}y^{\frac{1}{2}}}{2x^{\frac{1}{2}}} + \frac{\dot{y}x^{\frac{1}{2}}}{2y^{\frac{1}{2}}} = \frac{y\dot{x}+x\dot{y}}{2\sqrt{xy}}$ the Fluxion required.

Or thus.

Put $z = \sqrt{xy}$, then $zz = xy$, in Fluxions $2z\dot{z} = y\dot{x} + x\dot{y}$; therefore $\dot{z} = \frac{y\dot{x}+x\dot{y}}{2z} = \frac{y\dot{x}+x\dot{y}}{2\sqrt{xy}}$.

EXAM. 13.

Find the Fluxion of $\sqrt{xy-yy} = \overline{xy-yy}|^{\frac{1}{2}}$.

Then $\frac{1}{2} \times \overline{xy - yy}^{-\frac{1}{2}} \times \overline{x\dot{y} + y\dot{x} - 2y\dot{y}} = \frac{x\dot{y} + y\dot{x} - 2y\dot{y}}{2\sqrt{xy - yy}}$.

Or thus.

Put $z = \sqrt{xy - yy}$; then $zz = xy - yy$, in Fluxions $2z\dot{z} = x\dot{y} + y\dot{x} - 2y\dot{y}$, and $\dot{z} = \frac{x\dot{y} + y\dot{x} - 2y\dot{y}}{2z}$ $= \frac{x\dot{y} + y\dot{x} - 2y\dot{y}}{2\sqrt{xy - yy}}$.

EXAM. 14.

Required the Fluxion of $\frac{ax^2 + 2a^2x + a^3}{4x^3 - 3ax^2 + 3a^2x + a^3}$?

This muſt be managed in the ſame manner as to find the Fluxion of a fraction.

1. The Fluxion of the numerator is $= 2ax\dot{x} + 2a^2\dot{x}$.

2. Of the denominator is $= 12x^2\dot{x} - 6ax\dot{x} + 3a^2\dot{x}$.

Then $\overline{2ax\dot{x} + 2a^2\dot{x}} \times \overline{4x^3 - 3ax^2 + 3a^2x + x^3}$ ſubtracting from it $\overline{12x^2\dot{x} - 6ax\dot{x} + 3a^2\dot{x}} \times \overline{ax^2 + 2a^2x + a^3}$, is the Fluxion required, when divided by the ſquare of the denominator.

EXAM. 15.

Required the Fluxion of $\frac{\overline{a - bx + cx^2 - dx^3 + rx^4}}{s - tx + mx^2}$?

1. The Fluxion of the numerator is $= \overline{4rx^3\dot{x} - 3dx^2\dot{x} + 2cx\dot{x} - b\dot{x}} \times \overline{mx^2 - tx + s}$ ſubtracting from it $\overline{2mx\dot{x} - t\dot{x}} \times \overline{rx^4 - dx^3 + cx^2 - bx + a}$, and divide by the ſquare of the denominator.

EXAM. 16.

To find the Fluxion of $y^4 + az^3 - ay^2 + by + \frac{cx^3}{a + x} - xx\sqrt{ay + xx}$?

1. The Fluxion of $y^4 = 4y^3\dot{y}$.

2. —— of $az^3 = 3az^2\dot{z}$.

3. —— of $-ay^2 = -2ay\dot{y}$.

4. —— of $by = b\dot{y}$.

5. —— of $\frac{cx^3}{a+x} = \frac{3cx^2\dot{x}}{a+x} - \frac{cx^3\dot{x}}{\overline{a+x}|^2} = \frac{3acx^2\dot{x} + 2cx^3\dot{x}}{\overline{a+x}|^2}$.

6. —— of $xx\sqrt{ay+xx} = \frac{4ayx\dot{x} + 6x^3\dot{x} + ax^2\dot{y}}{2\sqrt{ay+xx}}$ by *Ex.* 9.

Now collect the Fluxions with their reſpective ſigns, and we ſhall have $4y^3\dot{y} + 3az^2\dot{z} - 2ay\dot{y} + b\dot{y} + \frac{3acx^2\dot{x} + 2cx^3\dot{x}}{aa+2ax+xx} - \frac{4ayx\dot{x} + 6x^3\dot{x} + ax^2\dot{y}}{2\sqrt{ay+xx}}$ for the Fluxion of $y^4 + az^3 - ay^2 + by + \frac{cx^3}{a+x} - xx\sqrt{ay+xx}$.

I ſhall here ſet down a Table of ſome general *Formulas*, or univerſal *Theorems*, by which quantities may be put into Fluxions by bare ſubſtitution.

FLUENTS.	FLUXIONS.
I. x^m	$mx^{m-1}\dot{x}$
II. ax^m	$max^{m-1}\dot{x}$
III. x^my^n	$y^nmx^{m-1}\dot{x} + x^mny^{n-1}\dot{y}$
IV. ax^my^n	$ay^nmx^{m-1}\dot{x} + ax^mny^{n-1}\dot{y}$
V. $x^my^nz^c$	$x^my^ncz^{c-1}\dot{z} + x^mz^cny^{n-1}\dot{y} + y^nz^cmx^{m-1}\dot{x}$
VI. $x^{m+1}y^{n+1}$	$\overline{n+1}.\dot{y}x^{m+1}y^n + \overline{m+1}.\dot{x}x^my^{n+1}$
VII. $\frac{1}{x^m}$	$-\frac{m\dot{x}}{x^{m+1}}$

FLUENTS.	FLUXIONS.
VIII. $\frac{1}{x^{m+1}y^{n+1}}$	$-\frac{\overline{n+1}.xy+\overline{m+1}.y\dot{x}}{x^{m+2}y^{n+2}}$
IX. $x^{\frac{1}{m}}$	$\frac{1}{m}x^{\frac{1}{m}-1}\dot{x}$
X. $x^{\frac{m}{n}}$ or $\sqrt[n]{x^m}$	$\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$
XI. $\frac{1}{x^{\frac{m}{n}}}$ or $\frac{1}{\sqrt[n]{x^m}}$	$-\frac{m}{n}x^{-\frac{m}{n}-1}\dot{x}$
XII. $\overline{x^m y^n}\vert^p$	$\overline{px^my^n}\vert^{p-1}y^nmx^{m-1}\dot{x}+x^mny^{n-1}\dot{y}$
XIII. $\frac{x^m}{y^n}$	$\frac{y^nmx^{m-1}\dot{x}-x^mny^{n-1}\dot{y}}{y^{2n}}$
XIV. $\sqrt[n]{x^my^c}$	$\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}y^{\frac{c}{n}}+\frac{c}{n}y^{\frac{c}{n}-1}\dot{y}x^{\frac{m}{n}}$

PROP. VII.

To find the Fluxion of Logarithms.

RULE.

Divide the Fluxion of the given number by the number itſelf, the quotient is the Fluxion of the logarithm of that number.

EXAMPLES.

1. The Fluxion of the logarithm of x is $\frac{\dot{x}}{x}$.

2. The Fluxion of the log. $1+x$ is $\frac{\dot{x}}{1+x}$.

For the Fluxion of $1 + x$ is $\dot{x}$.

3. The Fluxion of the log. $x + y$ is $\frac{\dot{x}+\dot{y}}{x+y}$.

4. The Fluxion of the log. $ax^3 + by^4$ is $\frac{3ax^2\dot{x} + 4by^3\dot{y}}{ax^3 + by^4}$.

PROP. VIII.

To find the Fluxions of the Powers of Logarithms.

RULE.

Proceed in the ſame manner as directed, *Prop.* II. to find a Fluxion, viz. multiply the number by the index, then leſſen the index by 1.

Or by the nature of Logarithms.

The Log. of the power of any number is equal to the Log. of the root multiplied by the index of that power.

EXAM. I.

Required the Fluxion of the Log. of xx?

1. 2 Log. $x^{2-1} = 2$ Log. x.

2. The Fluxion of the root Log. x is $\frac{\dot{x}}{x}$.

3. Then 2 Log. $x \times \frac{\dot{x}}{x} = \frac{2 \text{ Log. } x\dot{x}}{x}$ the Fluxion required.

COR.

1. The Fluxion of the Log. of x^m is $\frac{m\dot{x}}{x}$.

2. The Fluxion of the Log.$^m \dot{x}$ or L$^m \dot{x}$ is mLog.$^{m-1} x \times \frac{\dot{x}}{x}$.

3. The tabular logarithm of any number multiplied by 2.3025851 gives the hyperbolic logarithm of that number. Thus the hyp. log. of 8 is $2.3025851 \times \text{log. } 8 = 2.3025851 \times .90309 = 2.0794415$, the hyp. log. of 8, and ſo of any other.

EXAM. 2.

What is the Fluxion of the hyperbolic logarithm of $6 + x$*?*

1. The Fluxion of $6 + x$ is $= \dot{x}$.

2. Then $\frac{\dot{x}}{6+x} = \dot{x} \times \frac{1}{6+x}$ is the Fluxion of the log. $6 + x$.

3. But $\dot{x} \times \frac{1}{6+x} = \dot{x} \times \frac{1}{6} - \frac{x}{36} + \frac{x^2}{216} - \frac{x^3}{1296} + \frac{x^4}{7776}$ &c. by diviſion, $= \frac{\dot{x}}{6} - \frac{x\dot{x}}{36} + \frac{x^2\dot{x}}{216} - \frac{x^3\dot{x}}{1296} + \frac{x^4\dot{x}}{7776}$ &c.

4. And 2.3025851 log. $6 = 1.791756$.

5. Therefore $1.791756 + \frac{\dot{x}}{6} - \frac{x\dot{x}}{36} + \frac{x^2\dot{x}}{216} - \frac{x^3\dot{x}}{1296} + \frac{x^4\dot{x}}{7776}$ &c. is the Fluxion of the hyp. log. $6 + x$, and ſo of any other.

Alſo by the nature of logarithms we can find the Fluxions of *exponentials*, whoſe indices are variable and indeterminate.

RULE.

The logarithm of the root of any power, multiplied by the exponent or index of the power, is the logarithm of that power.

EXAMPLES.

1. That a^x in logarithms is x log. a.
2. That $\overline{a+b}|^x$ in log. is x log. $\overline{a+b}$.

Let X denote the hyperbolic logarithm of x.
Y the hyp. log. of y.
Z the hyp. log. of z.

EXAM. 1.

Required the Fluxion of $X^2 = y$.

Its Fluxion is $2X\dot{X} = \dot{y}$; but $\dot{X} = \frac{\dot{x}}{x}$, therefore $\dot{y} = \frac{2X\dot{x}}{x}$.

EXAM. 2.

Required the Fluxion of $X^4 = y$.

Its Fluxion is $4X^3\dot{X} = \dot{y}$, but $\dot{X} = \frac{\dot{x}}{x}$, therefore $\dot{y} = \frac{4X^3\dot{x}}{x}$.

EXAM. 3.

Required the Fluxion of a^x?

1. By the nature of logarithms, the log. of $a^x = x$ log. a; its Fluxion $= \dot{x}$ log. a.

2. Therefore a^x log. $a\dot{x}$ is the Fluxion required.

Or thus.

Put $z = a^x$, then log. $a^x =$ log. z, that is, x log. $a =$ log. z; in Fluxions $\dot{x}$ log. $a = \frac{\dot{z}}{z}$, and $\dot{z} = z\dot{x}$ log. a, therefore for z ſubſtitute its equal a^x, it will be $a^x\dot{x}$ log. a, or a^x log. $a\dot{x}$.

Corollary.

The Fluxion of $\overline{a+b}|^x$ will be $= \overline{a+b}|^x \, \dot{x} \times$ log. $\overline{a+b}$.

And of a^{x-1} is $= a^{x-1} \dot{x}$ log. a.

EXAM.

EXAM. 4.

Required the Fluxion of $z^x = y$.

Let $Z =$ hyperbolic log. of z, and Y the hyp. log. of y, then will $xZ = Y$, in Fluxions $x\dot{Z} + Z\dot{x} = \dot{Y}$, but $\dot{Z} = \frac{\dot{z}}{z}$, and $\dot{Y} = \frac{\dot{y}}{y}$ by the nature of log. for $\dot{Z}$ ſubſtitute its value $\frac{\dot{z}}{z}$ in the equation $x\dot{Z} + Z\dot{x} = \dot{Y}$, then $\frac{x\dot{z}}{z} + Z\dot{x} = \dot{Y} = \frac{\dot{y}}{y}$, therefore $\dot{y} = \frac{xy\dot{z}}{z} + Zy\dot{x} = xz^{x-1}\dot{z} + Zz^x\dot{x}$.

Or thus.

Put $y = z^x$, then log. $z^x =$ log. y; but the log. $z^x = x$ log. z, then x log. $z =$ log. y, in Fluxions log. $z\dot{x} + x$ log. $\dot{z} =$ log. $\dot{y}$, or log. $z\dot{x} + x\frac{\dot{z}}{z} = \frac{\dot{y}}{y}$, but $y = z^x$, then log. $z\dot{x} + x\frac{\dot{z}}{z} = \frac{\dot{y}}{z^x}$, therefore z^x log. $z\dot{x} + z^{x-1}\dot{x} = \dot{y}$, the Fluxion ſought.

Or thus.

The Fluxion of z^x is $z^{x+\dot{x}} + x\dot{z}z^{x+\dot{x}-1} - z^x$.

For the Fluxion of z^x is $= \overline{z + \dot{z}}|^{x+\dot{x}}$.

Then $z^x + \overline{z + \dot{z}}|^{x+\dot{x}} = z^{x+\dot{x}} + x\dot{z}z^{x+\dot{x}-1}$.

Therefore if the index be the ſum, product, or power of fluents, it is but ſubſtituting it, and its Fluxion inſtead of x and $\dot{x}$ in this.

OF

SECOND FLUXIONS.

WE have already obſerved, that the Fluxion of x is $\dot{x}$, of y is $\dot{y}$, &c. and at Art. 9, we ſhewed, that there were Fluxions of Fluxions, or an infinite variety of Fluxions, thus if $\dot{x}$ be the Fluxion of x,

Then $\ddot{x}$ is the ſecond Fluxion of x or Fluxion of $\dot{x}$.
$\dddot{x}$ the third Fluxion of x, or ſecond Flux. of $\dot{x}$.
$\ddddot{x}$ the fourth Fluxion of x.
&c.

This being obſerved, the ſecond Fluxions are eaſily determined from their firſt by the ſame rules with which the firſt Fluxions are found from finite Fluents. Yet when we thus proceed to ſecond, third, and following Fluxions, it is convenient to conſider ſome one quantity as flowing uniformly, and to write 1 for its firſt Fluxion, and 0 for the ſecond and following ones, by Art. 8. For inſtance; let the Fluxion of this Fluxion $\frac{y\dot{y}}{\dot{x}}$ be ſought. By ſuppoſing x to flow uniformly, and therefore $\dot{x}$ to be conſtant or $= 1$, the ſecond Fluxion will be $\frac{y\ddot{y}+\dot{y}^2}{\dot{x}}$.

COROLLARIES.

1. The Fluxion of $\dot{x}+\dot{y}$ is $\ddot{x}+\ddot{y}$, the ſecond Fluxion of $x+y$.

2. The Fluxion of $\ddot{x}+\ddot{y}$ is $\dddot{x}+\dddot{y}$, the third Fluxion of $x+y$.

3. The third Fluxion of $x+y-z+v$ will be $\dddot{x}+\dddot{y}-\dddot{z}+\dddot{v}$, which is the ſecond Fluxion of $\dot{x}+\dot{y}-\dot{z}+\dot{v}$; or the firſt Fluxion of $\ddot{x}+\ddot{y}-\ddot{z}+\ddot{v}$.

4. The Fluxion of xy has been found to be $x\dot{y} + y\dot{x}$, and the Fluxion of $x\dot{y} + y\dot{x}$ is $2\dot{x}\dot{y} + x\ddot{y} + y\ddot{x}$ the ſecond Fluxion of xy, when x and y are both variable.

5. The ſecond Fluxion of $\frac{x}{y}$ is

$$\frac{2x\dot{y}\dot{y} - 2y\dot{y}\dot{x} + yy\ddot{x} - xy\ddot{y}}{y^3}.$$

There is no difficulty in determining *ſecond*, *third*, &c. Fluxions, they are found in the ſame manner as the firſt. Thus the firſt Fluxion of xx is $2x\dot{x}$, and the Fluxion of $2x\dot{x}$ will be $2\dot{x}\dot{x} + 2x\ddot{x}$, or ſecond Fluxion of xx. For the Fluxion of x^2 is $2x\dot{x}$, and if $\dot{x}$ be ſuppoſed conſtant, that is, if the root x be generated with an equable and uniform velocity, the Fluxion of $2x\dot{x}$ or $2\dot{x} \times x$ will be (according to the rule $x^2 = 2x\dot{x}$) $2\dot{x} \times \dot{x}$ or $2\dot{x}\dot{x} = 2\dot{x}^2$. But if the velocity of x increaſe or decreaſe, then $\dot{x}$ expreſſing the meaſure of it will be variable, and muſt have its Fluxion. Thus x^2 is in Fluxions $2x\dot{x}$, then the Fluxion of $2\dot{x} \times x$ will be $2\dot{x}\dot{x} + 2x\ddot{x}$.

EXAM. I.

Required the ſecond Fluxion of x^3?

1. x^3 in Fluxions is $3x^2\dot{x}$ and $3\dot{x} \times x^2$ again in Fluxions $= 6x\dot{x} \times \dot{x} + 3x^2 \times \ddot{x} = 6x\dot{x}^2 + 3x^2\ddot{x}$ the ſecond Fluxion required.

EXAM. 2.

Required the ſecond Fluxion of x^4?

1. x^4 in Fluxions is $4x^3\dot{x}$, and $4\dot{x} \times x^3 = 4\dot{x} \times 3x^2\dot{x} = 12x^2\dot{x}^2$, and the Fluxion of $4\dot{x} \times x^3$ again will be $4x^3\ddot{x}$, therefore $12x^2\dot{x}\dot{x} + 4x^3\ddot{x}$ is the ſecond Fluxion of x^4.

Univerſally.

Let m denote the power of any variable quantity as x &c.

Then 1. $mx^{m-1}\dot{x}$ is the Fluxion of x^m or its firſt Fluxion.

2. $mm - mx^{m-2}\dot{x}\dot{x} + mx^{m-1}\ddot{x}$, the ſecond Fluxion of x^m.

3. $m^3 - 3m^2 + 2mx^{m-3}\dot{x}^3 + 3m^2 - 3mx^{m-2}\dot{x}\ddot{x} + mx^{m-1}\dddot{x}$, the third Fluxion of x^m.

&c. Alſo,

$$\overline{A+o}^{\frac{m}{n}} = A^{\frac{m}{n}} + \frac{m}{n}oA^{\frac{m-n}{n}} + \frac{m^2-mn}{2n^2}$$

$ooA^{\frac{m-2n}{n}}$ + &c. Now if any quantity $\overline{A+o}^{\frac{m}{n}}$ (where o is infinitely ſmall) be involved; the firſt, ſecond, third, &c. term, will be reſpectively as the flowing quantity $\left(A^{\frac{m}{n}}\right)$ the firſt, ſecond, &c. Fluxion of that quantity. Therefore $\frac{m^2-mn}{2n^2}ooA^{\frac{m-2n}{n}}$ is as the ſecond Fluxion of $A^{\frac{m}{n}}$. See page 43 of Mr. *Emerſon*'s Comment on Sir *Iſaac Newton's Principia*.

Thus to find the ſecond Fluxion of x^4. Here $m = 4$, by ſecond general above.

Then $mm - mx^{m-2}\dot{x}\dot{x} + mx^{m-1}\ddot{x} = 16 - 4x^2\dot{x}\dot{x} + 4x^3\ddot{x} = 12x^2\dot{x}\dot{x} + 4x^3\ddot{x}$ the ſame as above.

E X A M. 3.

Required the ſecond Fluxion of x^6?

Here $m = 6$; then $m^2 - mx^{m-2}\dot{x}\dot{x} + mx^{m-1}\ddot{x}$ is $= 36 - 6x^4\dot{x}\dot{x} + 6x^5\ddot{x} = 30x^4\dot{x}\dot{x} + 6x^5\ddot{x}$, the ſecond Fluxion of x^6.

And if $\dot{x}^2 = \dot{y}\dot{z}$, then $2\dot{x}\ddot{x} = \dot{z}\ddot{y} + \dot{y}\ddot{z}$, and ſo of others, the method being the ſame for all orders how high ſoever we go.

THE

INVERSE METHOD of FLUXIONS.

FLUENTS are found from their Fluxions by inſtituting operations the reverſe to thoſe by which Fluxions are found from Fluents.

The finding of Fluents is by far the moſt difficult, as well as the moſt uſeful part of the art of Fluxions, and this the noble inventor *Sir Iſaac Newton* was pleaſed to call, *Moleſtiſſimum et omnium difficillimum Problema*, i. e. the moſt difficult and troubleſome of all Problems.

PROP. IX.

Given an equation containing the Fluxions of quantities; to find the Fluents either in ſimple terms, or in an infinite ſeries.

1. To find the Fluent of any ſimple Fluxion.

RULE. I.

Set down the letters without the dots over them, they give their reſpective Fluents.

EXAM. I.

The Fluent of $\dot{x}$ *is* x. *The Fluent of* $\dot{y}$ *is* y. *Of* $\dot{z}$ *is* z. *Of* $\dot{v}$ *is* v.

EXAM. 2.

Let $\dot{x} = a\dot{z} + \dot{z}$. *Its Fluent is* $x = az + z$.

PROP. X.

To find the Fluent of any power of a variable quantity, multiplied by the Fluxion of the root.

RULE. 2.

1ſt, Expunge or ſtrike out the Fluxion of that quantity. 2dly, Add unity to the index of the power. 3dly, Divide the quantity thus raiſed by that index increaſed by unity, the quotient is the Fluent.

EXAM. 1.

Let $mx^{m-1}\dot{x} = \dot{y}$ *to find the Fluent.*

1. Strike out $\dot{x}$ the Fluxion, it is $= mx^{m-1}$.

2. Add unity to the index $m - 1$, is $= mx^{m-1+1} = mx^{m}$.

3. Divide by the new index m, it is $\frac{mx^m}{m} = x^m$ the Fluent required.

Therefore the Fluent of $mx^{m-1}\dot{x} = \dot{y}$ is $x^m = y$.

EXAM. 2.

Required the Fluent of $2x\dot{x} = \dot{y}$?

1. Strike out the Fluxion, it is $= 2x = 2x^1$.

2. Add unity to the index, is $= 2x^{1+1} = 2x^2$.

3. Divide by the increaſed index, it is $= \frac{2x^2}{2} = x^2$.

Therefore the Fluent of $2x\dot{x} = \dot{y}$ is $x^2 = y$.

EXAM. 3.

Required the Fluent of $3x^2\dot{x}$?

1. Strike out the Fluxion, it is $3x^2$.

2. Add 1 to the index, is $= 3x^3$.

3. Divide by index, it is $= \frac{3x^3}{3} = x^3$, the Fluent required.

EXAM.

EXAM. 4.

Required the Fluent of $35ax^4\dot{x}$?

1. Divide by Fluxion, it is $= 35ax^4$.
2. Add 1 to the index, is $= 35ax^5$.
3. Divide by index, is $= \frac{35ax^5}{5} = 7ax^5$, the Fluent ſought, as may be ſeen, *Prop.* II. in the direct method.

COROLLARIES.

1. The Fluent of $x^3\dot{x}$ is $= \frac{x^4}{4}$.
2. The Fluent of $2x^6\dot{x}$ is $= \frac{2x^7}{7}$.
3. The Fluent of $x\dot{x}$ is $= \frac{xx}{2}$.
4. The Fluent of $x^4\dot{x}$ is $= \frac{x^5}{5}$.

PROP. XI.

To find the Fluents of ſuch Fluxions, as involve two or more variable quantities.

RULE. 3.

1ſt, Multiply each term ſeparately by its variable quantity. 2dly, Divide it by its Fluxion and its new index. 3dly, Add all the terms together, and divide the ſum by the number of terms, gives the Fluent. N. B. This Rule only holds in ſome caſes.

EXAM. 1.

Required the Fluent of $y\dot{x} + x\dot{y}$?

$$\begin{array}{lll} \text{1. Mult.} & y\dot{x}, & x\dot{y} \\ \text{by} & x & y \\ \hline \text{Prod.} & yx\dot{x} & xy\dot{y}. \end{array}$$

2. Di-

2. Divide by the Fluxions $\dot{x}, \dot{y}$, the prod. is yx, xy.

3. Their ſum is $= yx + xy = 2xy$, which divided by 2, the number of terms, it is $\frac{2xy}{2} = xy$ the true Fluent.

EXAM. 2.

Required the Fluent of $ax\dot{y} + ay\dot{x}$?

$$\begin{array}{llll} \text{1. Mult.} & ax\dot{y}, & ay\dot{x} \\ \text{by} & y & x \\ \hline \text{Prod.} & axy\dot{y} & ayx\dot{x} \end{array}$$

2. Strike out the Fluxions, they become axy, ayx; their ſum, divided by the number of terms, is $\frac{axy + ayx}{2} = \frac{2axy}{2} = axy$ the Fluent.

EXAM. 3.

Required the Fluent of $yz\dot{x} + xz\dot{y} + xy\dot{z}$?

$$\begin{array}{llll} \text{1. Mult.} & yz\dot{x}, & xz\dot{y}, & xy\dot{z} \\ \text{by} & x & y & z \\ \hline \text{Prod.} & yzx\dot{x} & xzy\dot{y} & xyz\dot{z} \end{array}$$

2. Strike out the Fluxions, then prod. $= yzx$, xzy, xyz.

3. Their ſum, divided by number of terms, is $= \frac{3xyz}{3} = xyz$, the Fluent.

EXAM. 4.

Required the Fluent of $yzv\dot{x} + xzv\dot{y} + xyv\dot{z} + xyz\dot{v}$?

$$\begin{array}{lllll} \text{1. Mult.} & yzv\dot{x}, & xzv\dot{y}, & xyv\dot{z}, & xyz\dot{v}, \\ \text{by} & x & y & z & v \\ \hline \text{Prod.} & yzvx\dot{x} & xzvy\dot{y} & xyvz\dot{z} & xyzv\dot{v} \end{array}$$

2. Strike out the Fluxions, and divide the ſum by the number of terms, we have $\frac{yzvx + xzvy + xyvz + xyzv}{4} = \frac{4xyzv}{4} = xyzv$ the true Fluent.

EXAM

EXAM. 5.

Required the Fluent of $2y^3z^4v^5x\dot{x} + 3x^2z^4v^5y^2\dot{y} + 4x^2y^3v^5z^3\dot{z} + 5x^2y^3z^4v^4\dot{v}$?

1 Mul. $2y^3z^4v^5xx$, $3x^2z^4v^5y^2\dot{y}$, $4x^2y^3v^5z^3\dot{z}$, $5x^2y^3z^4v^4\dot{v}$.

by x, y, z, v

Prod. $2y^3z^4v^5x^2\dot{x}$, $3x^2z^4v^5y^3\dot{y}$, $4x^2y^3v^5z^4\dot{z}$, $5x^2y^3z^4v^5\dot{v}$.

2. Expunge the Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, out of the product, and it becomes $2y^3z^4v^5x^2$, $3x^2z^4v^5y^3$, $4x^2y^3v^5z^4$, $5x^2y^3z^4v^5$.

3. Divide by the indices thus increaſed, and we have,

4. $\frac{2y^3z^4v^5x^2}{2} = y^3z^4v^5x^2$, here the index of x is 2.

5. $\frac{3x^2z^4v^5y^3}{3} = x^2z^4v^5y^3$ — of y is 3.

6. $\frac{4x^2y^3v^5z^4}{4} = x^2y^3v^5z^4$ — of z is 4.

7. $\frac{5x^2y^3z^4v^5}{5} = x^2y^3z^4v^5$ — of v is 5.

8. Their ſum $y^3z^4v^5x^2 + x^2z^4v^5y^3 + x^2y^3v^5z^4 + x^2y^3z^4v^5 = 4x^2y^3z^4v^5$.

9. Divide by number of terms is $= \frac{4x^2y^3z^4v^5}{4}$ $= x^2y^3z^4v^5$ the Fluent required. See *Prop.* III. *the direct method of Fluxions.*

EXAM. 6.

To find the Fluent of $\frac{y\dot{x} - x\dot{y}}{yy}$, *or* $\frac{\dot{x}}{y} - \frac{x\dot{y}}{yy}$?

1. $\frac{y\dot{x} - x\dot{y}}{yy} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{yy} = y^{-1}\dot{x} - xy^{-2}\dot{y}$.

2. Then by proceeding as the above, the Fluent is $xy^{-1} = \frac{x}{y}$.

EXAM. 7.

To find the Fluent of $-mx^{-m-1}\dot{x}$?

1. Expunge $\dot{x}$ and add 1 to the index, it is then $-mx^{-m}$.

2. Divide by index, it will be $\frac{-mx^{-m}}{-m} = x^{-m}$ $= \frac{1}{x^m}$ the Fluent.

EXAM. 8.

Required the Fluent of $-\frac{ma\dot{x}}{x^{m+1}}$?

1. Expunge $\dot{x}$, it is $-\frac{ma}{x^{m+1}}$.

2. Add 1 to the index of the numerator, or rather take 1 from the index of the denominator, it is $= -\frac{ma}{x^m}$.

3. Divide by the index, it will be $= \frac{a}{x^m}$ the Fluent.

EXAM. 9.

Required the Fluent of $\frac{a\dot{x}}{x^m}$, *or* $a\dot{x}x^{-m}$?

1. Strike out $\dot{x}$ it is $= ax^{-m}$. Add 1, it is ax^{1-m}; divide by the index, it will be $= \frac{ax^{1-m}}{1-m}$ the Fluent.

Corollaries.

1. The Fluent of $\overline{a+y}|^4 \times \dot{y}$ is $= \frac{\overline{a+y}|^5}{5}$.

2. — of $\overline{a^m+x^m}|^n \times x^{m-1}\dot{x}$ is $= \frac{\overline{a^m+x^m}|^{n+1}}{m.\overline{n+1}}$.

3. — of $ax^m\dot{x}$ is $= \frac{ax^{m+1}}{m+1}$.

PROP.

PROP. XII.

If any term be divided by the first power of the variable quantity, as $\frac{\dot{x}}{x}$, $\frac{a\dot{x}}{x}$, $\frac{\dot{y}}{y}$, *&c. then the Fluent of that term must be found by itself by this*

RULE 4.

Multiply the given coefficient and the number 2.302585 into the logarithm of that variable quantity for the Fluent of that term.

EXAM. 1.

What is the Fluent of $\frac{\dot{x}}{x}$*?*

Now 2.302585 log. x, is the Fluent required.

EXAM. 2.

Required the Fluent of $\frac{a\dot{x}}{x}$*?*

Now 2.302585 $a \times$ log. x, is the Fluent.

EXAM. 3.

Required the Fluent of $\frac{-\dot{x}}{x}$*?*

Now $-$ 2.302585 $\times$ log. x, is the Fluent.

PROP. XIII.

To find the Fluents of Fluxions where surds are concerned.

RULE 5.

Reduce the radical quantities into simple terms by involution, except such where the fluxionary part is in a given ratio to the Fluxion of the root, and the roots of affected equations must be extracted, then proceed as in *Rule* 2.

EXAM.

EXAM. 1.

To find the Fluent of $x^{\frac{m}{n}}\dot{x}$.

1. Expunge $\dot{x}$, it will be $= x^{\frac{m}{n}}$.

2. Add 1 to index, is $= x^{\frac{m}{n}+1} = x^{\frac{m+n}{n}}$.

3. Divide by new index, it is $= \frac{x^{\frac{m+n}{n}}}{\frac{m+n}{n}} = \frac{nx^{\frac{m+n}{n}}}{m+n}$ the Fluent.

This is general, and may be illuſtrated thus.

EXAM. 2.

Required the Fluent of $\dot{x}\sqrt{x}$ *or* $x^{\frac{1}{2}}\dot{x}$?

Here $m = 1$, $n = 2$; then $\frac{nx^{\frac{m+n}{n}}}{m+n} = \frac{2x^{\frac{3}{2}}}{3}$ the Fluent.

Or thus, by *Rule* 2.

1. Strike out $\dot{x}$, it will be $= x^{\frac{1}{2}}$.
2. Add 1 to index, it is $= x^{\frac{1}{2}+1} = x^{\frac{3}{2}}$.
3. Divide by index $\frac{3}{2}$, it will be $= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2x^{\frac{3}{2}}}{3}$, as above.

EXAM. 3. *Univerſally.*

Required the Fluent of $\frac{1}{n}x^{\frac{1}{n}-1}\dot{x}$?

1. Strike out $\dot{x}$, it will be $= \frac{1}{n}x^{\frac{1}{n}-1}$.

2. Add 1 to index, it is $= \frac{1}{n}x^{\frac{1}{n}}$.

3. Divide by index, then $\dfrac{\frac{1}{n}x^{\frac{1}{n}}}{\frac{1}{n}} = x^{\frac{1}{n}}$ the Fluent.

COROLLARIES.

1. The Fluent of $\frac{m}{n} x^{\frac{m-n}{n}} \dot{x}$ is $= x^{\frac{m}{n}}$.

2. The Fluent of $-\frac{m}{n} x^{-\frac{m}{n}-1} \dot{x}$ is $= x^{-\frac{m}{n}}$

$= \dfrac{1}{x^{\frac{m}{n}}}$.

And by proceeding as in the laſt example, may all or moſt of the Fluents be found from their Fluxions in the foregoing Table, where the Fluents are given to find their Fluxions, which are general, and are uſeful ſeveral ways.

EXAM. 4.

Required the Fluent of $\dfrac{\dot{x}}{\sqrt{2x}}$ *or* $\dfrac{\dot{x}}{2x^{\frac{1}{2}}}$?

1. $\dfrac{\dot{x}}{2x^{\frac{1}{2}}} = \frac{1}{2} \times \dfrac{\dot{x}}{\sqrt{x}} = \frac{1}{2}\dot{x} \times x^{-\frac{1}{2}} = \frac{1}{2}\dot{x}x^{\frac{1}{2}-1}$, by the notation of exponents, *Sect.* I.

2. Now expunge $\dot{x}$, it will be $= \frac{1}{2}x^{\frac{1}{2}-1}$.

3. Add 1 to index, it becomes $= \frac{1}{2}x^{\frac{1}{2}}$.

4. Divide by index, it is $= \dfrac{\frac{1}{2}x^{\frac{1}{2}}}{\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$ the Fluent.

Or thus, by *Ex.* 3.

Here $\frac{1}{n} = \frac{1}{2}$.

Then $\dfrac{\frac{1}{n}x^{\frac{1}{n}}}{\frac{1}{n}} = x^{\frac{1}{n}} = x^{\frac{1}{2}}$ or $\sqrt{x}$.

Or thus, by COR. 1.

Here $m = 1$, $n = 2$.

Then $x^{\frac{m}{n}} = x^{\frac{1}{2}}$ or $\sqrt{x}$.

EXAM. 5.

What is the Fluent of $\dfrac{2x\dot{x}}{3x|^{3}}$ or $\dfrac{2}{3}xx^{-\frac{1}{3}}\dot{x}$?

1. Strike out the Fluxion, it will be $= \frac{2}{3}xx^{-\frac{1}{3}}$.
2. Add 1 to index, it is $= \frac{2}{3}xx^{-\frac{1}{3}+1} = \frac{2}{3}xx^{\frac{2}{3}}$.
3. Divide by the index, it is $= \dfrac{\frac{2}{3}xx^{\frac{2}{3}}}{\frac{2}{3}} = xx^{\frac{2}{3}} = \sqrt[3]{xx}$ the Fluent.

EXAM. 6.

Required the Fluent of $\sqrt{a}\sqrt{x^3} \times \dot{x}$ *or* $a^{\frac{1}{2}}x^{\frac{3}{2}}\dot{x}$?

1. Strike out $\dot{x}$, it becomes $a^{\frac{1}{2}}x^{\frac{3}{2}}$.
2. Add 1 to the index, is $= a^{\frac{1}{2}}x^{\frac{3}{2}+1} = a^{\frac{1}{2}}x^{\frac{5}{2}}$.
3. Divide by the index, it is $= \dfrac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{\frac{5}{2}} = \dfrac{2a^{\frac{1}{2}}x^{\frac{5}{2}}}{5}$ the Fluent.

EXAM. 7.

What is the Fluent of $\sqrt[3]{a} \times \sqrt{x^3} \times \dot{x}$, *or* $a^{\frac{1}{3}}x^{\frac{3}{2}}\dot{x}$?

1. Expunge $\dot{x}$, is $= a^{\frac{1}{3}}x^{\frac{3}{2}}$.
2. Add 1 to index, is $= a^{\frac{1}{3}}x^{\frac{5}{2}}$.
3. Divide by index, it will be $= \dfrac{a^{\frac{1}{3}}x^{\frac{5}{2}}}{\frac{5}{2}} = \dfrac{2}{5}a^{\frac{1}{3}}x^{\frac{5}{2}}$ the Fluent.

EXAM. 8.

Required the Fluent of $ab^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} - ab^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$?

1. Expunge $\dot{x}$, it is $= ab^{\frac{3}{2}}x^{-\frac{1}{2}} - ab^{\frac{1}{2}}x^{\frac{1}{2}}$.
2. Add 1 to the index $= ab^{\frac{3}{2}}x^{\frac{1}{2}} - ab^{\frac{1}{2}}x^{\frac{3}{2}}$.
3. Divide by the index, $= \dfrac{ab^{\frac{3}{2}}x^{\frac{1}{2}}}{\frac{1}{2}} - \dfrac{ab^{\frac{1}{2}}x^{\frac{3}{2}}}{\frac{3}{2}} =$

$2ab^{\frac{3}{2}}x^{\frac{1}{2}} - \dfrac{2ab^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$ the Fluent required.

RULE 6.

But *Fluents* are much eaſier found from Mr. *Emerſon*'s elegant Table by a bare ſubſtitution only. This method of finding *Fluents* being ſo eaſy and general, I hope the giving examples in ſome of the principal caſes will be both uſeful and acceptable to my Readers. Here it muſt be obſerved, that,

In comparing a given Fluxion with the Table, the radical *quantity muſt be compared with the* radical *quantity*; *the* index *of the variable quantity within the* radical, *with the* index *within the* radical; *and the* index *without, with the* index *without the* radical, &c. *ſo that by making equations between like quantities, the particular values of each may be diſcovered, in order to find the* Fluent *required.*

In order to ſhew the uſe of the Table which exhibits the eaſieſt and readieſt methods of ſolving this moſt difficult Problem for finding Fluents, I ſhall ſet down ſeveral caſes derived from the 3d, 4th, 5th, 6th, 7th, 8th, 9th, and 10th Forms, and diſpoſe them into a ſort of Table, ſetting down the Fluxions and their correſpondent Fluents over againſt them, and then ſhew their uſe by a few Examples.

Though I call them *Caſes* under their reſpective Forms, yet they are no more than particular examples deduced from Mr. *Emerſon*'s general Table, and the Reader would do well to have the book before him, when he is uſing theſe *Caſes*, which, when underſtood, he will be able to deduce others for his uſe.

But before the Table is conſulted, it will be proper to premiſe, that Mr. *Emerſon* has given rules at pages 104, &c. how to reduce a given fluxionary expreſſion to another, which he calls *reducing the index*. By theſe ſorts of reduction, a given Fluxion is prepared for a ſolution, when it does not fall directly at firſt under any of the forms. As for inſtance in the following examples.

1. $\frac{a\dot{x}}{\sqrt{2ax+xx}}$ is reduced to $\frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a+x}}$.

2. $$\frac{a\dot{x}}{\sqrt{2ax-xx}} = \frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}.$$

3. $$\frac{aa\dot{x}}{x\sqrt{aa+xx}} = \frac{aax^{-2}\dot{x}}{\sqrt{1+ax^{-2}}}.$$

4. $$\frac{a\dot{x}}{x}\sqrt{2ax+xx} = ax^{\frac{1}{2}-1}\dot{x}\sqrt{2a+x}.$$

5. $$\frac{abb\dot{x}}{\overline{bb-xx}\sqrt{aa-xx}} = \frac{abbx^{-3}\dot{x}}{\overline{-1+bbx^{-2}}\sqrt{-1+aax^{-2}}};$$

6. $$\frac{x^2\dot{x}}{\sqrt{aa-xx}} = \overline{-1+aax^{-2}}|^{-\frac{1}{2}}x\dot{x}.$$

7. $$\frac{x^{-1}\dot{x}}{aa\sqrt{ab+cx-xx}} = \frac{x^{-2}\dot{x}}{aa\sqrt{-1+cx^{-1}+abx^{-2}}},$$

8. $$\frac{\dot{x}\sqrt{bb-xx}}{aa+xx} = \frac{x^{-1}\dot{x}\sqrt{-1+bbx^{-2}}}{1+aax^{-2}} = \frac{bb\dot{x}-x^2\dot{x}}{\overline{aa+xx}\sqrt{bb-xx}}.$$

9. $$\frac{-x^2\dot{x}}{\overline{aa+xx}\sqrt{bb-xx}} = \frac{-\dot{x}}{\sqrt{bb-xx}} + \frac{aa\dot{x}}{\overline{aa+xx}\sqrt{bb-xx}}.$$

10. $$\frac{a\dot{x}}{2x}\sqrt{\frac{a+3x}{x}} = \frac{ax^{-\frac{3}{2}}\dot{x}}{2}\sqrt{a+3x}.$$

11. $\frac{aa\dot{x}\sqrt{bb-xx}}{xx\times aa+xx} = \frac{-aax^{-3-1}\dot{x}\sqrt{bb-xx}}{1+aax^{-2}}.$

12. $\frac{ax^{\frac{1}{3}}\dot{x}}{\overline{b-cx^{\frac{2}{3}}}|^{\frac{3}{5}}} = \overline{b-cx^{\frac{2}{3}}}|^{-\frac{3}{5}} \times ax^{\frac{1}{3}}\dot{x} = -$
$\overline{c+bx^{-\frac{2}{3}}}|^{-\frac{3}{5}} \times ax^{-\frac{1}{15}}\dot{x}.$

13. $\overline{a+cz^{n}}|^{m}.a\dot{z}z^{sn-1} = \overline{c+az^{-n}}|^{m}.a\dot{z}z^{nm+sn-1}.$

14. $\dot{x}x^{-6}\times\overline{a^{2}+x^{2}}|^{\frac{1}{2}} = \dot{x}x^{-6}\times\overline{1+a^{2}x^{-2}}|^{\frac{1}{2}}.$

15. $\frac{aa\dot{y}\sqrt{cc-yy}}{yy\times aa+yy}$ is $= \frac{aay^{-2-1}\dot{y}\sqrt{cc-yy}}{1+aay^{-2}}.$

THE

THE TABLE.

FORM III.

Cases.	FLUXIONS.	FLUENTS.
1	$\overline{\alpha+\beta z^\eta}\vert^{-\frac{1}{2}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{1}{2}}}{\frac{1}{2}\eta\beta}$
2	$\overline{\alpha+\beta z^\eta}\vert^{\frac{1}{2}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{3}{2}}}{\frac{3}{2}\eta\beta}$
3	$\overline{\alpha+\beta z^\eta}\vert^{\frac{3}{2}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{5}{2}}}{\frac{5}{2}\eta\beta}$
4	$\overline{\alpha+\beta z^\eta}\vert^{-\frac{1}{3}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{2}{3}}}{\frac{2}{3}\eta\beta}$
5	$\overline{\alpha+\beta z^\eta}\vert^{\frac{1}{3}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{4}{3}}}{\frac{4}{3}\eta\beta}$
6	$\overline{\alpha+\beta z^\eta}\vert^{\frac{2}{3}} z^{\eta-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^\eta}\vert^{\frac{5}{3}}}{\frac{5}{3}\eta\beta}$
7	$\overline{\alpha+\beta z^{-2}}\vert^{\mu} z^{-1}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^{-2}}\vert^{\mu+1}}{\overline{\mu+1}\times-2\beta}$
8	$\overline{\alpha+\beta z^{-1}}\vert^{\mu} z^{-2}\dot{z}$	$\dfrac{\overline{\alpha+\beta z^{-1}}\vert^{\mu+1}}{\overline{\mu+1}\times-\beta}$
9	$\overline{\alpha+\beta z}\vert^{\mu}\dot{z}$	$\dfrac{\overline{\alpha+\beta z}\vert^{\mu+1}}{\overline{\mu+1}\,\beta}$
10	$\overline{\alpha+\beta.z^2}\vert^{\mu} z\dot{z}$	$\dfrac{\overline{\alpha+\beta z^2}\vert^{\mu+1}}{\overline{\mu+1}.2\beta}$
11	$\overline{\alpha+\beta z^3}\vert^{\mu} z^2\dot{z}$ &c.	$\dfrac{\overline{\alpha+\beta z^3}\vert^{\mu+1}}{\overline{\mu+1}.3\beta}$ &c.

FORM

FORM IV.

Caſes.	FLUXIONS.	FLUENTS.
1	$\frac{z^{-3}\dot{z}}{\alpha+\beta z^{-2}}$	$\frac{2.30258}{-2\beta} \times \log. \overline{\alpha+\beta z^{-2}}$
2	$\frac{z^{-2}\dot{z}}{\alpha+\beta z^{-1}}$	$\frac{2.30258}{-\beta} \times \log. \overline{\alpha+\beta z^{-1}}$
3	$\frac{\dot{z}}{\alpha+\beta z}$	$\frac{2.30258}{\beta} \times \log. \overline{\alpha+\beta z}$
4	$\frac{z\dot{z}}{\alpha+\beta z^{2}}$	$\frac{2.30258}{2\beta} \times \log. \overline{\alpha+\beta z^{2}}$
5	$\frac{z^{2}\dot{z}}{\alpha+\beta z^{3}}$	$\frac{2.30258}{3\beta} \times \log. \overline{\alpha+\beta z^{3}}$
	&c.	&c.

FORM V.

Caſes.	FLUX.	FLUENTS.
1	$\frac{z^{-2}\dot{z}}{\alpha+\beta z^{-2}}$	$\frac{.017453}{-\sqrt{\alpha\beta}} \times$ deg. in the arch whoſe tangt. $= \sqrt{\frac{\beta}{azz}}$, rad. $= 1$
2	$\frac{z^{-\frac{3}{2}}\dot{z}}{\alpha+\beta z^{-1}}$	$\frac{.017453}{-\frac{1}{2}\sqrt{\alpha\beta}} \times$ deg. tang. $= \sqrt{\frac{\beta}{\alpha z}}$, rad. $= 1$
3	$\frac{z^{-\frac{1}{2}}\dot{z}}{\alpha+\beta z}$	$\frac{.017453}{\frac{1}{2}\sqrt{\alpha\beta}} \times$ deg. tang. $= \sqrt{\frac{\beta z}{\alpha}}$, rad. $= 1$
4	$\frac{\dot{z}}{\alpha+\beta z^{2}}$	$\frac{.017453}{\sqrt{\alpha\beta}} \times$ deg. tang. $= z\sqrt{\frac{\beta}{\alpha}}$, rad. $= 1$
5	$\frac{z^{\frac{1}{2}}\dot{z}}{\alpha+\beta z^{3}}$	$\frac{.017453}{\frac{3}{2}\sqrt{\alpha\beta}} \times$ deg. tang. $= z\sqrt{\frac{\beta z}{\alpha}}$, rad. $= 1$
	&c.	&c.

FORM VI.

Cases.	Flux.	Fluents.
1	$\frac{z^{-2}\dot{z}}{\alpha-\beta z^{-2}}$	$\frac{2.30258}{-2\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^{-2}}}{\sqrt{\alpha}-\sqrt{\beta z^{-2}}}$
2	$\frac{z^{-\frac{3}{2}}\dot{z}}{\alpha-\beta z^{-1}}$	$\frac{2.30258}{-\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^{-1}}}{\sqrt{\alpha}-\sqrt{\beta z^{-1}}}$
3	$\frac{z^{-\frac{1}{2}}\dot{z}}{\alpha-\beta z}$	$\frac{2.30258}{\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z}}{\sqrt{\alpha}-\sqrt{\beta z}}$
4	$\frac{\dot{z}}{\alpha-\beta z^{2}}$	$\frac{2.30258}{2\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^{2}}}{\sqrt{\alpha}-\sqrt{\beta z^{2}}}$
5	$\frac{z^{\frac{1}{2}}\dot{z}}{\alpha-\beta z^{3}}$	$\frac{2.30258}{3\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^{3}}}{\sqrt{\alpha}-\sqrt{\beta z^{3}}}$
	&c.	&c.

FORM IX.

Cases.	Fluxions.	Fluents.
1	$\frac{z^{-2}\dot{z}}{\sqrt{\alpha+\beta z^{-2}}}$	$\frac{2.30258}{-\sqrt{\beta}} \times \log. \sqrt{\beta z^{-2}} + \sqrt{\alpha+\beta z^{-2}}$
2	$\frac{z^{-\frac{3}{2}}\dot{z}}{\sqrt{\alpha+\beta z^{-1}}}$	$\frac{2.30258}{-\frac{1}{2}\sqrt{\beta}} \times \log. \sqrt{\beta z^{-1}} + \sqrt{\alpha+\beta z^{-1}}$
3	$\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{\alpha+\beta z}}$	$\frac{2.30258}{\frac{1}{2}\sqrt{\beta}} \times \log. \sqrt{\beta z} + \sqrt{\alpha+\beta z}$
4	$\frac{\dot{z}}{\sqrt{\alpha+\beta z^{2}}}$	$\frac{2.30258}{\sqrt{\beta}} \times \log. \sqrt{\beta z^{2}} + \sqrt{\alpha+\beta z^{2}}$
5	$\frac{z^{\frac{1}{2}}\dot{z}}{\sqrt{\alpha+\beta z^{3}}}$	$\frac{2.30258}{\frac{3}{2}\sqrt{\beta}} \times \log. \sqrt{\beta z^{3}} + \sqrt{\alpha+\beta z^{3}}$
6	$\frac{z\dot{z}}{\sqrt{\alpha+\beta z^{4}}}$	$\frac{2.30258}{2\sqrt{\beta}} \times \log. \sqrt{\beta z^{4}} + \sqrt{\alpha+\beta z^{4}}$
	&c.	&c.

FORM X.

Cases.	FLUXIONS.	FLUENTS.
1	$\frac{z^{-2}\dot{z}}{\sqrt{\alpha-\beta z^{-2}}}$	$\frac{.017453}{-\sqrt{\beta}}\times$ deg. in the arch whose sine $= \frac{1}{z}\sqrt{\frac{\beta}{\alpha}}$, rad. $=1$
2	$\frac{z^{-\frac{3}{2}}\dot{z}}{\sqrt{\alpha-\beta z^{-1}}}$	$\frac{.017453}{-\sqrt{\beta}}\times$ deg. sine $= \frac{2}{\alpha}\sqrt{\frac{\alpha\beta z-\beta^2}{zz}}$, rad. $=1$
3	$\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{\alpha-\beta z}}$	$\frac{.017453}{\sqrt{\beta}}\times$ deg. sine $= \frac{2}{\alpha}\sqrt{\alpha\beta z-\beta^2 z^2}$, rad. $=1$
4	$\frac{\dot{z}}{\sqrt{\alpha-\beta z^2}}$	$\frac{.017453}{\sqrt{\beta}}\times$ deg. sine $= z\sqrt{\frac{\beta}{\alpha}}$, rad. $=1$
5	$\frac{z^{\frac{1}{2}}\dot{z}}{\sqrt{\alpha-\beta z^3}}$	$\frac{.017453}{3\sqrt{\beta}}\times$ deg. sine $= \frac{2}{\alpha}\sqrt{\alpha\beta z^3-\beta^2 z^6}$, rad $=1$
6	$\frac{z\dot{z}}{\sqrt{\alpha-\beta z^4}}$	$\frac{.017453}{2\sqrt{\beta}}\times$ deg. sine $= zz\sqrt{\frac{\beta}{\alpha}}$, rad. $=1$
	&c.	&c.

N. B. Theſe Tables I publiſhed in my *Miſcellanea Curioſa Mathematica*, but as that book long ſince is out of print, I have, at the deſire of ſome friends, republiſhed them here; the uſe of which I come now to ſhew.

EXAM. I.

What is the Fluent of $\frac{a\dot{x}}{\sqrt{a-x}}$, *or* $\overline{a-x}|^{-\frac{1}{2}}\times a\dot{x}$ *?*

This belongs to Caſe 1, Form 3, where we have $\alpha=a$, $\beta=-1$, $z=x$, $\eta=1$. Then the Fluent of $\overline{a-x}|^{-\frac{1}{2}}\times a\dot{x}$ is $=\left(\frac{\overline{\alpha+\beta z^{\eta}}|^{\frac{1}{2}}}{\frac{1}{2}\eta\beta}=\right)\frac{\overline{a-x}|^{\frac{1}{2}}}{\frac{1}{2}\times-1}=$

$-2 \times \sqrt{a - x}$. And Fluent of $\frac{a\dot{x}}{\sqrt{a - x}} = -2a\sqrt{a - x}$.

EXAM. 2.

Required the Fluent of $2x\dot{x}\sqrt{aa + xx}$, *or* $\overline{a^2 + x^2}|^{\frac{1}{2}} \times 2x\dot{x}$?

By Caſe 2, Form 3, we have $\alpha = aa$, $\beta = 1$, $z = x$, $n = 2$. And Fluent of $\overline{aa + xx}|^{\frac{1}{2}} \times 2x\dot{x} =$

$$\left(\frac{\overline{\alpha + \beta z^n}|^{\frac{3}{2}}}{\frac{3}{2}n\beta} =\right)\frac{\overline{2.a^2 + x^2}|^{\frac{3}{2}}}{\frac{3}{2} \times 2} = \frac{\overline{a^2 + x^2}|^{\frac{3}{2}}}{\frac{3}{2}} = \frac{\overline{2.a^2 + x^2}|^{\frac{3}{2}}}{3}$$

$$= \frac{2aa + 2xx}{3}\sqrt{aa + xx}.$$

EXAM. 3.

Required the Fluent of $\frac{x\dot{x}}{\sqrt{aa + xx}}$, *or* $\overline{aa + xx}|^{-\frac{1}{2}} \times x\dot{x}$?

By Caſe 1, Form 3, we have $\alpha = aa$, $\beta = 1$, $z = x$, $n = 2$. And the Fluent of $\overline{aa + xx}|^{-\frac{1}{2}} \times x\dot{x}$ is $= \frac{\overline{aa + xx}|^{\frac{1}{2}}}{\frac{1}{2} \times 2} = \overline{aa + xx}|^{\frac{1}{2}} = \sqrt{aa + xx}$.

EXAM. 4.

To find the Fluent of $\frac{-x\dot{x}}{\sqrt{aa - xx}}$ *or* $\overline{aa - xx}|^{-\frac{1}{2}} \times -x\dot{x}$?

By Caſe 1, Form 3, we have $\alpha = aa$, $\beta = -1$, $z = x$, $n = 2$. And Fluent of $\overline{aa - xx}|^{-\frac{1}{2}} \times -x\dot{x}$ is $= \frac{\overline{aa - xx}|^{\frac{1}{2}}}{\frac{1}{2} \times 2} = \overline{aa - xx}|^{\frac{1}{2}} = \sqrt{aa - xx}$.

EXAM. 5.

To find the Fluent of $dx^2\dot{x}\sqrt{aa - bbx^3} - x\dot{x}\sqrt[3]{aa + bbx^2} + \frac{a\dot{y}}{\sqrt{a - y}} - \frac{2by\dot{y}}{\sqrt{aa - yy}}$?

1. For the *Fluent* $d\sqrt{aa - bbx^3} \times x\dot{x}$; by Caſe 11, Form 3, we have $\alpha = aa$, $\beta = -bb$, $z = x$, $\mu = \frac{1}{2}$, whence *Fluent* $\overline{a^2 - b^2x^3}|^{\frac{1}{2}}x^2\dot{x} = \left(\frac{\overline{\alpha+\beta z^3}|^{\mu+1}}{\mu+1.3\beta} =\right) \frac{\overline{aa-bbx^3}|^{\frac{3}{2}}}{\frac{3}{2}\times -3bb} = \frac{\overline{aa-bbx^3}|}{-\frac{9bb}{2}} = -\frac{2}{9bb} \times \overline{aa-bbx^3}|^{\frac{3}{2}}$; and the *Fluent* of $d\sqrt{aa-bbx^3} \times x^2\dot{x} = -\frac{2d}{9bb} \times \overline{aa-bbx^3}|^{\frac{3}{2}}$.

2. For *Fluent* $x\dot{x}\sqrt[3]{aa+bbx^2} = \overline{aa+bbxx}|^{\frac{1}{3}} \times x\dot{x}$. By Caſe 10, Form 3, $\alpha = aa$, $\beta = bb$, $z = x$, $\mu = \frac{1}{3}$, and *Fluent* $\overline{aa+bbxx}|^{\frac{1}{3}}x\dot{x} = \left(\frac{\overline{\alpha+\beta z^2}|^{\mu+1}}{\mu+1.2\beta} =\right) \frac{\overline{aa+bbxx}|^{\frac{4}{3}}}{\frac{4}{3}\times 2bb} = \frac{3}{8} \times \frac{\overline{aa+bbxx}|^{\frac{4}{3}}}{bb}$; and Fl. $-\overline{aa+bbxx}|^{\frac{1}{3}}x\dot{x} = -\frac{3}{8} \times \frac{\overline{aa+bbxx}|^{\frac{4}{3}}}{bb}$.

3. For *Fluent* $\frac{a\dot{y}}{\sqrt{a-y}}$ or $\overline{a-y}|^{-\frac{1}{2}} \times a\dot{y}$. By Caſe 1, Form 3, the *Fluent* of $\overline{a-y}|^{-\frac{1}{2}}\dot{y} = \frac{\overline{a-y}|^{\frac{1}{2}}}{\frac{1}{2}\times -1} = -2 \times \sqrt{a-y}$, and Fl. $\frac{a\dot{y}}{\sqrt{a-y}} = -2a\sqrt{a-y}$.

4. For *Fluent* $-\frac{2by\dot{y}}{\sqrt{aa-yy}}$; now $\frac{y\dot{y}}{\sqrt{aa-yy}} = \overline{aa-yy}|^{-\frac{1}{2}}y\dot{y}$; by Caſe 1, Form 3, $\alpha = aa$, $\beta = -1$, $z = y$, $n = 2$, then Fl. $\overline{aa-yy}|^{-\frac{1}{2}}y\dot{y} = \left(\frac{\overline{\alpha+\beta z^n}|^{\frac{1}{2}}}{\frac{1}{2}n\beta} =\right) \frac{\overline{aa-yy}|^{\frac{1}{2}}}{\frac{1}{2}\times 2 \times -1} = \frac{\overline{aa-yy}|^{\frac{1}{2}}}{\frac{1}{2}\times -2} = -\sqrt{aa-yy}$, and mul-

tiplying by $-2b$, Fl. $\frac{-2by\dot{y}}{\sqrt{aa-yy}} = 2b\sqrt{aa-yy}$, whence the whole *Fluent* required is $-\frac{2d}{9bb} \times \overline{aa-bbx^3}|^{\frac{3}{2}} - \frac{3}{8} \times \frac{\overline{aa+bbxx}|^{\frac{4}{3}}}{bb} - 2a\sqrt{a-y} + 2b\sqrt{aa-yy}$.

EXAM. 6.

To find the Fluent of $\frac{2m}{a} \times y\dot{y}\sqrt{aa+4yy}$?

By Caſe 10, Form 3, we have $\alpha = aa$, $\beta = 4$, $z = y$, $\mu = \frac{1}{2}$, the *Fluent* $\overline{aa+4yy}|^{\frac{1}{2}}y\dot{y} = \frac{\overline{aa+4yy}|^{\frac{3}{2}}}{\frac{3}{2} \times 8} = \frac{\overline{aa+4yy}|^{\frac{3}{2}}}{\frac{24}{2}} = \frac{\overline{aa+4yy}|^{\frac{3}{2}}}{12}$; the Fl. $\frac{2m}{a} \times y\dot{y}\sqrt{aa+yy} = \frac{2m}{12a} \times \overline{aa+4yy}|^{\frac{3}{2}} = \frac{m}{6a} \times \overline{aa+4yy}|^{\frac{3}{2}}. = \frac{m}{6a} \times \overline{aa+4yy}\sqrt{aa+4yy}$.

EXAM. 7.

Required the Fluent of $x\dot{x}\sqrt{\overline{a+x}|^3}$ *or* $\overline{a+x}|^{\frac{3}{2}}.x\dot{x}$?

By Caſe 3, Form 3, $\alpha = a$, $\beta = 1$, $z = x$, $n = 1$, whence Fl. $\overline{a+x}|^{\frac{3}{2}}.x\dot{x} = \frac{\overline{a+x}|^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2.\overline{a+x}|^{\frac{5}{2}}}{5}$.

EXAM. 8.

To find the Fluent of $\frac{x\dot{x}}{\sqrt[3]{aa-xx}}$ *or* $\overline{aa-xx}|^{-\frac{1}{3}}x\dot{x}$?

By Caſe 4, Form 3, $\alpha = aa$, $\beta = -1$, $z = x$, $n = 2$, whence Fl. $\overline{aa-xx}|^{-\frac{1}{3}}x\dot{x}, = \frac{\overline{aa-xx}|^{\frac{2}{3}}}{\frac{2}{3} \times 2} = \frac{3.\overline{aa-xx}|^{\frac{2}{3}}}{4}$.

EXAM. 9.

To find the Fluent of $\frac{\dot{x}}{a+x}$?

By Caſe 3, Form 4, we have $\alpha = a$, $\beta = 1$, $z = x$, whence the Fluent of $\frac{\dot{x}}{a+x} = \left(\frac{\dot{z}}{\alpha+\beta z} =\right)$ $2.30258 \times \log. \overline{a+x}$. That is, the Fluent of $\frac{\dot{x}}{a+x}$ is the hyperbolic logarithm of $\overline{a+x}$.

EXAM. 10.

To find the Fluent of $\frac{4a^2x\dot{x}}{a^3b+b^2x^2} - \frac{8\dot{y}}{b-ay}$.

By Caſe 4, Form 4, $\alpha = a^3b$, $\beta = b^2$, $z = x$, and *Fluent* $\frac{x\dot{x}}{a^3b+b^2x^2} = \frac{2.30258}{2bb} \times \log. \overline{a^3b+b^2x^2}$, and $4aa \times \frac{x\dot{x}}{a^3b+b^2x^2} = \frac{2aa \times 2.30258}{bb} \times \log.$ $\overline{a^3b+b^2x^2}$.

2. For $\frac{-8\dot{y}}{b-ay}$; by Caſe 3, Form 4, $\alpha = b$, $\beta = -a$, $z = y$, and Fluent of $\frac{\dot{y}}{b-ay} = \frac{2.30258}{-a} \times$ $\log. \overline{b-ay}$; and *Fluent* $\frac{-8\dot{y}}{b-ay} = 2.30258 \times \frac{8}{a} \times$ $\log. \overline{b-ay}$. Therefore the *Fluent* required is $2.30258 \times \log. \overline{a^3b+b^2x^2} + 2.30258 \times \frac{8}{a} \times \log.$ $\overline{b-ay}$; or, if you pleaſe, the Fluent required is $= \frac{2aa}{bb} \times$ hyperbolic log. of $a^3b+b^2x^2 + \frac{8}{a} \times$ hyp. log. $\overline{b-ay}$.

EXAM. 11.

To find the Fluent of $\frac{x\dot{x}}{a^2+x^2}$.

By Form 4, we have $\alpha = aa$, $\beta = 1$, $z = x$, $n = 2$, whence Fl. $\frac{x\dot{x}}{aa+xx} = \left(\frac{z\dot{z}}{\alpha+\beta z^n} =\right) \frac{2.30258}{2} \times \log.\overline{aa+xx}$.

EXAM. 12.

To find the Fluent of $\frac{6\dot{x}}{aa+xx}$.

Compare this with Caſe 4, Form 5, $\alpha = aa$, $\beta = 1$, $z = x$, then Fl. $\frac{\dot{x}}{aa+xx} = \left(\frac{\dot{z}}{\alpha+\beta z^2} =\right) \frac{.017453}{a} \times$ degrees whoſe natural tangent is $x\sqrt{\frac{1}{aa}} = \frac{x}{a}$. Therefore the Fluent of $\frac{6\dot{x}}{aa+xx} = \frac{.017453 \times 6}{a} \times$ degrees in the arch whoſe tangent is $\frac{x}{a}$.

Now if you deſire to work it into numbers, ſuppoſe, for example, that $a = 20$, $x = 43$, then $\frac{x}{a} = 2.15$ a tangent; look for it in the Table of *natural tangents*, and you will find it belong to 65 degrees and 3 min. nearly, that is, to ſomething more than $65\frac{3}{60}$ degrees. And to find the degrees exactly, ſay,

As the difference of the two neareſt tangents in the Table (2.1510378 — 2.1494020 =)	.0016358
To the difference between the leſſer and given tangent (2.15 — 2.1494020 =)	.0005980
So is 1 minute —	1.0000000
To a proportional part to be added	.3657

Therefore the number of degrees belonging to the tangent 2.15 is $65\frac{3.3657}{60}$ or 65.05609; therefore the Fluent of $\frac{6\dot{x}}{aa+xx}$ is $= \frac{.017453}{a} \times 65.05609$ $= \frac{6 \times .017453 \times 65\ 05609}{20} = .34063$.

But if you will suppose x infinite; then $\frac{x}{a}$ or $\frac{x}{20}$ will be infinite, and the degrees belonging to the infinite tangent will be 90°, and therefore in this caſe the Fluent of $\frac{6\dot{x}}{aa+xx}$ will be $\frac{6 \times .01745 \times 90}{20}$ $= .47124$.

EXAM. 13.

To find the Fluent of $\frac{13\dot{x}}{aa+xx}$.

By Caſe 4, Form 5, we have $\alpha = aa$, $\beta = 1$, $z = x$, whence Fl. $\frac{\dot{x}}{aa+xx} = \frac{.017453}{a} \times$ degrees in the arch whoſe tangent $= \frac{x^2}{a}$; therefore the Fluent of $\frac{13\dot{x}}{aa+xx} = \frac{13 \times .017453}{a} \times$ deg. in the arch whoſe natural tangent is $\frac{xx}{a}$, rad. $= 1$.

EXAM. 14.

To find the Fluent of $\frac{\dot{x}\sqrt{x}}{aa+6x^3}$, *or* $\frac{x^{\frac{1}{2}}\dot{x}}{aa+6x^3}$.

By Caſe 5, Form 5, we have $\alpha = aa$, $\beta = 6$, $z = x$, whence Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{a^2+x^3} = \frac{.017453}{\frac{3}{2}a\sqrt{6}} \times$ degrees in the arch, whoſe natural tangent is $= x\sqrt{\frac{6x}{aa}} = \frac{2\times.017453}{3a\sqrt{6}} \times$ deg. in the arch, whoſe tangent $= \frac{x}{a}\sqrt{6x}$.

EXAM. 15.

Let the Fluxion propoſed be $\frac{a\dot{x}}{ax^{\frac{1}{2}}+6x^{\frac{3}{2}}}$, *or* $\frac{ax^{-\frac{1}{2}}\dot{x}}{a+6x}$.

By Caſe 3, Form 5, we have $\alpha = a$, $\beta = 6$, $z = x$, then Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a+6x} = \frac{.017453}{\frac{1}{2}\sqrt{6a}} \times$ degrees in the arch, whoſe tangent $= \sqrt{\frac{6x}{a}} = \frac{2\times.017453}{\sqrt{6a}} \times$ degrees in the arch, whoſe tangent is $\sqrt{\frac{6x}{a}}$.

EXAM. 16.

To find the Fluent of $\frac{7ax\dot{x}}{c^4+x^4}$.

By Form 5, $z = x$, $n = 4$, $\alpha = c^4$, $\beta = 1$, and *Fluent* $\frac{x\dot{x}}{c^4+x^4} = \frac{2\times.017453}{2\sqrt{c^4}} \times$ degrees whoſe tangent is $\sqrt{\frac{x^4}{c^4}} = \frac{.017453}{cc} \times$ deg. whoſe tangent is $\frac{xx}{cc}$; and *Fluent* $\frac{7ax\dot{x}}{c^4+x^4} = \frac{7a\times.017453}{cc} \times$ degrees in the arch, whoſe tangent is $\frac{xx}{cc}$.

EXAM. 17.

To find the Fluent of $\frac{c\dot{x}\sqrt{x}}{ab^3+cx^3}$.

By Caſe 5, Form 5, we have $\alpha = ab^3$, $\beta = c$, $z = x$,

$z = x$, then Fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{ab^3 + cx^3}$ = to $\frac{.017453}{\frac{3}{2} \times \sqrt{ab^3 \times c}}$ $= \frac{2 \times .017453}{3\sqrt{ab^3c}}$ × degrees whoſe tangent is $x\sqrt{\frac{cx}{ab^3}}$; and Fl. $\frac{c\dot{x}\sqrt{x}}{ab^3 + cx^3} = \frac{2c \times .017453}{3\sqrt{ab^3c}}$ × degrees in the arch, whoſe tangent is $x\sqrt{\frac{cx}{ab^3}} = \frac{2 \times .017453}{3b\sqrt{abc}}$ × degrees in the arch, whoſe tangent is $\frac{x}{b}\sqrt{\frac{cx}{ab}}$.

EXAM. 18.

To find the Fluent of $\frac{11x^{-2}\dot{x}}{1 + aax^{-2}}$ *or* $\frac{11\dot{x}}{xx \times \overline{aa + xx}}$.

Here, by Form 5, Caſe 1, the Fluent of $\frac{x^{-2}\dot{x}}{1 + aax^{-2}} = \frac{.017453}{-\sqrt{aa}}$ × deg. whoſe tangent is $\sqrt{\frac{aax^{-2}}{1}}$ or $\frac{a}{x}$; and Fluent of $\frac{11x^{-2}\dot{x}}{1 + aax^{-2}} = \frac{11 \times .017453}{a}$ × deg. in the arch whoſe tangent is $\frac{a}{x}$.

EXAM. 19.

To find the Fluent of $\frac{\dot{x}}{a - x^2}$.

Comparing this with Caſe 4, Form 6, we have $\alpha = a$, $\beta = 1$, $z = x$, $(\eta = 2)$; whence Fl. $\frac{\dot{x}}{a - xx}$ $= \frac{2.30258}{2\sqrt{a}} \times \log. \frac{\sqrt{a} + x}{\sqrt{a} - x}$.

EXAM. 20.

To find the Fluent of $\frac{6\dot{x}}{aa - xx}$.

Comparing this with Caſe 4, Form 6, we have $\alpha = aa$, $\beta = 1$, $z = x$, ($\eta = 2$ in Mr. *Emerſon*'s Table);

Table): and *Fluent* $\frac{\dot{x}}{aa-xx}=\frac{2.30258}{2\sqrt{aa}}\times\log.\frac{\sqrt{aa}+\sqrt{xx}}{\sqrt{aa}-\sqrt{xx}}=\frac{2.30258}{2a}\times\log.\frac{a+x}{a-x}$, and Fluent $\frac{6\dot{x}}{aa-xx}=\frac{6\times2.30258}{2a}\times\log.\frac{a+x}{a-x}$.

Or Fl. $\frac{\dot{x}}{aa-xx}=\left(\frac{\mathrm{L}}{n\sqrt{\alpha\beta}}\times\log.\ \alpha+\beta z^n+2\sqrt{\alpha\beta z^n}=\right)\frac{2.30258}{2a}\times\log.\frac{aa+xx+2\sqrt{a^2x^2}}{aa-xx}$.

Therefore the Fluent $\frac{6\dot{x}}{aa-xx}=\frac{3\times2.30258}{a}\times\log.\frac{aa+xx-2ax}{aa-xx}$.

Or *Fluent* $\frac{\dot{x}}{aa-xx}=\left(\frac{\mathrm{L}}{n\sqrt{\alpha\beta}}\times\log.\frac{\sqrt{\beta z^n}+\sqrt{\alpha}}{\sqrt{\beta z^n}-\sqrt{\alpha}}\right.=\frac{2.30258}{2a}\times\log.\frac{x+a}{x-a}$; and *Fluent* $\frac{6\dot{x}}{aa-xx}=\frac{3\times2.30258}{a}\times\log.\frac{x+a}{x-a}$. Or, laſtly, *Fluent* $\frac{\dot{x}}{aa-xx}=\left(\frac{\mathrm{L}}{n\sqrt{\alpha\beta}}\times\log.\frac{\alpha-\beta z^n}{\alpha+\beta z^n-2\sqrt{\alpha\beta z^n}}=\right)\frac{2.30258}{2a}\times\log.\frac{aa-xx}{aa+xx-2\sqrt{a^2x^2}}$, and *Fluent* $\frac{6\dot{x}}{aa-xx}=\frac{3\times2.30258}{a}\times\log.\frac{aa-xx}{aa+xx-2ax}$.

So that the Fluent may be expreſſed any of theſe ſeveral ways. Now ſuppoſe you want to know the value of the Fluent in numbers; let $a=20$, $x=43$; then as x is greater than a, we will take the Fluent $=\frac{3\times2.30258}{a}\times\log.\frac{x+a}{x-a}$, and $\frac{x+a}{x-a}=\frac{63}{23}=2.73913$; ſeek for this number in the Table of logarithms, and you will find its log. = 0.4376127; therefore the Fluent of $\frac{6\dot{x}}{aa-xx}$, or which is the ſame, of $\frac{-6\dot{x}}{xx-aa}$, is $\frac{3\times 2.30258}{20}\times$ .4376127.

Or if you ſuppoſe $a = 20$, and $x =$ infinity, then $\frac{x+a}{x-a} = 1$, whoſe log. is 0, and then the Fluent of $\frac{-6\dot{x}}{xx-aa} = \frac{3\times 2.30258}{20}\times 0 = 0$.

But if you will ſuppoſe $a = 20$, $\dot{x} = 12$, then $\frac{a+x}{a-x} = \frac{32}{8} = 4$, whoſe logarithm is 0.60206, and then the Fluent $\frac{6\dot{x}}{aa-xx} = \frac{6\times 2.30258}{2a}\times 0.60206$ $= 2.30258\times\frac{3\times 0.60206}{20}$.

And if you ſuppoſe $a = 20$, $x = 20$, then $\frac{x+a}{x-a} = \frac{40}{0} =$ infinity, whoſe logarithm is infinite; therefore in this caſe Fl. $\frac{6\dot{x}}{aa-xx} = \frac{6\times 2.30258}{2a}\times$ infinity = infinity.

EXAM. 21.

To find the Fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{a^2-5x^3}$.

Comparing this with Caſe 5, Form 6, we have $\alpha = aa$, $\beta = 5$, $z = x$, ($n = 2$ in the original Table) whence Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{a^2-5x^3} = \frac{2.30258}{3\sqrt{5aa}}\times$ log. $\frac{\sqrt{aa}+\sqrt{5x^3}}{\sqrt{aa}-\sqrt{5x^3}} = \frac{2.30258}{3a\sqrt{5}}\times$ log. $\frac{a+x\sqrt{5x}}{a-x\sqrt{5x}}$

EXAM.

EXAM. 22.

To find the Fluent of $\frac{abb}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{\overline{aa-bb}-bbv}$.

Comparing this with Caſe 3, Form 6, where $a = aa - bb$, $\beta = bb$, $z = v$, then $\frac{2.30258}{\sqrt{\overline{aa-bb} \times bb}} \times \log. \frac{\sqrt{aa-bb}+\sqrt{bbv}}{\sqrt{aa-bb}-\sqrt{bbv}} = \frac{2.30258}{b\sqrt{aa-bb}} \times \log. \frac{\sqrt{aa-bb}+b\sqrt{v}}{\sqrt{aa-bb}-b\sqrt{v}}$. And $\frac{abb}{2} \times \frac{2.30258}{b\sqrt{aa-bb}} \times \log. \frac{\sqrt{aa-bb}+b\sqrt{v}}{\sqrt{aa-bb}-b\sqrt{v}}$ is the Fluent required.

EXAM. 23.

To find the Fluent of $\frac{\dot{y}}{y} = \frac{ma\dot{x} \pm \overline{m+n}.x\dot{x}}{\overline{m+n}.ax \pm xx}$.

The Fluent of the firſt ſide, viz. $\frac{\dot{y}}{y}$ is by Form 2, $= 2.30258 \times \log. y$. The ſecond ſide is found by Form 4, but it muſt firſt be divided into two parts $\frac{ma\dot{x}}{\overline{m+n}.ax \pm xx} \pm \frac{\overline{m+n}.x\dot{x}}{\overline{m+n}.ax \pm xx}$. Divide the firſt quantity by xx, and we have $\frac{ma\dot{x}}{\overline{m+n}.ax \pm x^2} = \frac{ma}{m+n} \times \frac{x^{-2}\dot{x}}{\pm 1 + ax^{-1}}$; here by Form 4, $\alpha = \pm 1$, $\beta = a$, $v = -1$, and the Fluent of $\frac{x^{-2}\dot{x}}{\pm 1 + ax^{-1}} = \frac{2.30258}{-a} \times \log. \overline{\pm 1 + ax^{-1}} = \frac{-2.30258}{a} \times \log. \frac{a \pm x}{x}$.

Again, for the Fluent of $\pm\frac{m+n}{m+n}\times\frac{x\dot{x}}{ax\pm xx}$, or $\frac{\dot{x}}{a\pm x}$, where $n=1$, $\alpha=a$, $\beta=\pm 1$, then by Form 4, *Fluent* $\frac{\dot{x}}{a\pm x}=\frac{2.30258}{\pm 1}\times\log.\ \overline{a\pm x}$; hence therefore the Fluent of the proposed Fluxion is $2.30258\times\log.\ y=\frac{-2.30258m}{m+n}\times\log.\ \frac{a\pm x}{x}+2.30258\times\log.\ a\pm x$, or dividing by 2.30258, $\log.\ y=\frac{-m}{m+n}\times\log.\ \frac{a\pm x}{x}+\log.\ a\pm x$, as required.

EXAM. 24.

To find the Fluent of $y\dot{x}=20x\dot{x}+x\dot{x}\sqrt{400-xx}$.

The Fluent of $20x\dot{x}$ is $=\frac{20xx}{2}=10x^2$; and Fluent $x\dot{x}\sqrt{400-xx}$ is to be compared with Case 2, Form 3, where $\alpha=400$, $\beta=1$, $x=z$, $n=2$, then Fl. $\overline{400-xx}|^{\frac{1}{2}}x\dot{x}=\frac{\overline{400-xx}|^{\frac{3}{2}}}{\frac{3}{2}\times 2}=\frac{\overline{400-xx}|^{\frac{3}{2}}}{3}$; therefore the whole Fluent is $yx=10x^2+\frac{\overline{400-xx}|^{\frac{3}{2}}}{3}$.

EXAM. 25.

To find the Fluent of $\frac{\dot{x}\sqrt{8a-3x}}{\overline{2a-x}|^{\frac{3}{2}}}$, *or* $\frac{\dot{x}\times\overline{8a-3x}|^{\frac{1}{2}}}{\overline{2a-x}|^{\frac{3}{2}}}$.

Compare it with Form 24, Mr. *Emerſon*'s Table, and we have $e=8a$, $f=-3$, $z=x$, $\theta=1$, $m=\frac{1}{2}$, $\lambda=1$, $g=2a$, $b=-1$, $r=1$, $p=-6a+8a=2a$, $v=\frac{8a-3x}{2a-x}$, therefore the Fluxion

ion will be transformed into $\frac{\overline{-3+v}|^{-1}v^{\frac{1}{2}}\dot{v}}{1} = \frac{v^{\frac{1}{2}}\dot{v}}{v-3}$ $= v^{-\frac{1}{2}}\dot{v} - \frac{3v^{-\frac{1}{2}}\dot{v}}{3-v}$, whoſe Fluent by Form 1 and 6, is $2v^{\frac{1}{2}} - \frac{3\times 2.30258}{\sqrt{3}} \times \log. \frac{3+v+2\sqrt{3v}}{3-v}$, therefore the Fluent required is $2a\sqrt{v} - 2.30258a\sqrt{3}$ $+ \log. \frac{3+v+2\sqrt{3v}}{3-v}$, in which x may be reſtored.

EXAM. 26.

To find the Fluent of $\frac{ax^{-\frac{1}{2}}\dot{x}}{a-6x}$, *or* $\frac{a\dot{x}}{ax^{\frac{1}{2}}-6x^{\frac{3}{2}}}$.

If this be compared with Caſe 3, Form 6, we have $\alpha = a$, $\beta = 6$, $z = x$; then Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} =$ $\frac{2.30258}{\sqrt{6a}} \times \log. \frac{\sqrt{a}+\sqrt{6x}}{\sqrt{a}-\sqrt{6x}}$, when a is greater than $6x$.

Or Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\beta z^n}+\sqrt{\alpha}}{\sqrt{\beta z^n}-\sqrt{\alpha}} = \right)$ $\frac{2.30258}{\sqrt{6a}} \times \log. \frac{\sqrt{6x}+\sqrt{a}}{\sqrt{6x}-\sqrt{a}}$ for the Fluent, when $6x$ is greater than a.

Or Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\alpha+\beta z^n+2\sqrt{\alpha\beta z^n}}{\alpha-\beta z^n}\right.$ $\left. = \right) \frac{2.30258}{\sqrt{6a}} \times \log. \frac{a+6x+2\sqrt{6ax}}{a-6x}$, the Fluent when a is greater than $6x$.

Or, laſtly, Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log.\right.$ $\left.\frac{\alpha-\beta z^n}{\alpha+\beta z^n-2\sqrt{\alpha\beta z^n}} = \right) \frac{2.30258}{\sqrt{6a}} \times \log.$

$a - 6x$

$\frac{a-6x}{a+6x-2\sqrt{6ax}}$, being the Fluent if a is greater than $6x$, then multiply any of theſe by a, you have the Fluent required.

EXAM. 27.

Let the Fluent of $\frac{7a\dot{x}x}{c^4-x^4}$ *be required.*

By Form 6, Caſe 4, we have $\alpha=c^4$, $\beta=1$, $\eta=4$, in the original Table, and $z=x$, then the Fluent of $\frac{x\dot{x}}{c^4-x^4}=\left(\frac{L}{\eta\sqrt{\alpha\beta}}\times\text{log.}\ \frac{\sqrt{\alpha}+\sqrt{\beta z^{\eta}}}{\sqrt{\alpha}-\sqrt{\beta z^{\eta}}}=\right)$ $\frac{2.30258}{4cc}\times\text{log.}\ \frac{cc+xx}{cc-xx}$, the Fluent when cc is greater than xx.

Or Fl. $\frac{x\dot{x}}{c^4-x^4}$; or Fl. $\frac{-x\dot{x}}{c^4-x^4}=\left(\frac{L}{\eta\sqrt{\alpha\beta}}\times\text{log.}\right.$ $\left.\frac{\sqrt{\beta z^{\eta}}+\sqrt{\alpha}}{\sqrt{\beta z^{\eta}}-\sqrt{\alpha}}=\right)\frac{2.30258}{4cc}\times\text{log.}\ \frac{xx+cc}{xx-cc}$, being the Fluent when xx is greater than cc.

Therefore the Fluent of $\frac{7ax\dot{x}}{c^4-x^4}$ is $=\frac{7a\times 2.30258}{4cc}$ $\times\text{log.}\ \frac{cc+xx}{cc-xx}$, or $=\frac{7a\times 2.30258}{4cc}\times\text{log.}\ \frac{xx+cc}{xx-cc}$.

EXAM. 28.

To find the Fluent of $\frac{c\dot{x}\sqrt{x}}{ab^3-cx^3}$.

Comparing this with Form 6, there is $z=x$, $\eta=3$, $\alpha=ab^3$, $\beta=c$, and φ, or Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{ab^3-cx^3}=$ $\left(\frac{L}{\eta\sqrt{\alpha\beta}}\times\text{log.}\ \frac{\sqrt{\alpha}+\sqrt{\beta z^{\eta}}}{\sqrt{\alpha}-\sqrt{\beta z^{\eta}}}=\right)\frac{2.30258}{3\sqrt{ab^3c}}\times\text{log.}$ $\frac{\sqrt{ab^3}+\sqrt{cx^3}}{\sqrt{ab^3}-\sqrt{cx^3}}$, when ab^3 is greater than cx^3.

ion will be transformed into $\frac{\overline{-3+v}|^{-1}v^{\frac{1}{2}}\dot{v}}{1} = \frac{v^{\frac{1}{2}}\dot{v}}{v-3}$ $= v^{-\frac{1}{2}}\dot{v} - \frac{3v^{-\frac{1}{2}}\dot{v}}{3-v}$, whoſe Fluent by Form 1 and 6, is $2v^{\frac{1}{2}} - \frac{3\times 2.30258}{\sqrt{3}} \times \log. \frac{3+v+2\sqrt{3v}}{3-v}$, therefore the Fluent required is $2a\sqrt{v} - 2.30258a\sqrt{3}$ $+ \log. \frac{3+v+2\sqrt{3v}}{3-v}$, in which x may be reſtored.

E x a m. 26.

To find the Fluent of $\frac{ax^{-\frac{1}{2}}\dot{x}}{a-6x}$, *or* $\frac{a\dot{x}}{ax^{\frac{1}{2}}-6x^{\frac{3}{2}}}$.

If this be compared with Caſe 3, Form 6, we have $\alpha = a$, $\beta = 6$, $z = x$; then Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} =$ $\frac{2.30258}{\sqrt{6a}} \times \log. \frac{\sqrt{a}+\sqrt{6x}}{\sqrt{a}-\sqrt{6x}}$, when a is greater than $6x$.

Or Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = (\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\beta z^n}+\sqrt{\alpha}}{\sqrt{\beta z^n}-\sqrt{\alpha}} =)$ $\frac{2.30258}{\sqrt{6a}} \times \log. \frac{\sqrt{6x}+\sqrt{a}}{\sqrt{6x}-\sqrt{a}}$ for the Fluent, when $6x$ is greater than a.

Or Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = (\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\alpha+\beta z^n+2\sqrt{\alpha\beta z^n}}{\alpha-\beta z^n}$ $=)\ \frac{2.30258}{\sqrt{6a}} \times \log. \frac{a+6x+2\sqrt{6ax}}{a-6x}$, the Fluent when a is greater than $6x$.

Or, laſtly, Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{a-6x} = (\frac{L}{n\sqrt{\alpha\beta}} \times \log.$ $\frac{\alpha-\beta z^n}{\alpha+\beta z^n-2\sqrt{\alpha\beta z^n}} =)\ \frac{2.30258}{\sqrt{6a}} \times \log.$

$a-6x$

$\frac{a-6x}{a+6x-2\sqrt{6ax}}$, being the Fluent if a is greater than $6x$, then multiply any of theſe by a, you have the Fluent required.

EXAM. 27.

Let the Fluent of $\frac{7ax\dot{x}}{c^4-x^4}$ *be required.*

By Form 6, Caſe 4, we have $\alpha = c^4$, $\beta = 1$, $n = 4$, in the original Table, and $z = x$, then the Fluent of $\frac{x\dot{x}}{c^4-x^4} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^n}}{\sqrt{\alpha}-\sqrt{\beta z^n}} =\right)$ $\frac{2.30258}{4cc} \times \log. \frac{cc+xx}{cc-xx}$, the Fluent when cc is greater than xx.

Or Fl. $\frac{x\dot{x}}{c^4-x^4}$, or Fl. $\frac{-x\dot{x}}{c^4-x^4} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\beta z^n}+\sqrt{\alpha}}{\sqrt{\beta z^n}-\sqrt{\alpha}} =\right)$ $\frac{2.30258}{4cc} \times \log. \frac{xx+cc}{xx-cc}$, being the Fluent when xx is greater than cc.

Therefore the Fluent of $\frac{7ax\dot{x}}{c^4-x^4}$ is $= \frac{7a\times 2.30258}{4cc}$ $\times \log. \frac{cc+xx}{cc-xx}$, or $= \frac{7a\times 2.30258}{4cc} \times \log. \frac{xx+cc}{xx-cc}$.

EXAM. 28.

To find the Fluent of $\frac{c\dot{x}\sqrt{x}}{ab^3-cx^3}$.

Comparing this with Form 6, there is $z = x$, $n = 3$, $\alpha = ab^3$, $\beta = c$, and φ, or Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{ab^3-cx^3} =$ $\left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\alpha}+\sqrt{\beta z^n}}{\sqrt{\alpha}-\sqrt{\beta z^n}} =\right)$ $\frac{2.30258}{3\sqrt{ab^3c}} \times \log.$ $\frac{\sqrt{ab^3}+\sqrt{cx^3}}{\sqrt{ab^3}-\sqrt{cx^3}}$, when ab^3 is greater than cx^3.

Or φ, or Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{ab^3 - cx^3} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\beta z^n} + \sqrt{\alpha}}{\sqrt{\beta z^n} - \sqrt{\alpha}} = \right) \frac{2.30258}{3b\sqrt{abc}} \times \log. \frac{x\sqrt{cx} + b\sqrt{ab}}{x\sqrt{cx} - b\sqrt{ab}}$, when cx^3 is greater than ab^3.

Therefore Fl. $\frac{c\dot{x}\sqrt{x}}{ab^3 - cx^3} = \frac{2.30258c}{3b\sqrt{abc}} \times \log. \frac{b\sqrt{ab} + x\sqrt{cx}}{b\sqrt{ab} - x\sqrt{cx}}$, or $= \frac{2.30258c}{3b\sqrt{abc}} \times \log. \frac{x\sqrt{cx} + b\sqrt{ab}}{x\sqrt{cx} - b\sqrt{ab}}$.

EXAM. 29.

To find the Fluent of $\frac{11y^{-2}\dot{y}}{1 - aay^{-2}}$, *or* $\frac{11\dot{y}}{yy \times aa - yy}$.

Comparing this with Form 6, $z = y$, $n = -2$, $\alpha = 1$, $\beta = aa$, and φ, or Fl. $\frac{y^{-2}\dot{y}}{1 - aay^{-2}} = \left(\frac{L}{n\sqrt{\alpha\beta}}\right.$

$$\left.\times \log. \frac{\sqrt{\alpha} + \sqrt{\alpha\beta z^n}}{\sqrt{\alpha} - \sqrt{\alpha\beta z^n}}, = \right) \frac{2.30258}{-2a} \times \log. \frac{1 + \frac{a}{y}}{1 - \frac{a}{y}}$$

$$= \frac{2.30258}{-2a} \times \log. \frac{y+a}{y-a},$$ when y is greater than a.

Or Fl. $\frac{y^{-2}\dot{y}}{1 - aay^{-2}} = \left(\frac{L}{n\sqrt{\alpha\beta}} \times \log. \frac{\sqrt{\beta z^n} + \sqrt{\alpha}}{\sqrt{\beta z^n} - \sqrt{\alpha}} = \right)$

$$\frac{2.30258}{-2a} \times \log. \frac{\frac{a}{y} + 1}{\frac{a}{y} - 1} = \frac{2.30258}{-2a} \times \log. \frac{a+y}{a-y},$$

when a is greater than y.

Therefore Fl. $\frac{11y^{-2}\dot{y}}{1 - aay^{-2}} = \frac{11 \times 2.30258}{-2a} \times \log. \frac{y+a}{y-a}$, or $= \frac{11 \times 2.30258}{-2a} \times \log. \frac{a+y}{a-y}$.

EXAM. 30.

To find the Fluent of $\frac{\dot{z}}{a^3 + z^3}$.

Compare this with Form 7, and we have $z = z$, $\eta = 3$, $\alpha = a^3$, $\beta = 1$; $\frac{\delta}{\lambda} = \frac{1}{3}$; $K = \frac{180^\circ}{3} = 60^\circ$, $x = \sqrt[3]{\frac{z^3}{a^3}} = \frac{z}{a}$, $a = s.\ 60^\circ = \sqrt{\tfrac{3}{4}}$, $b = 0$, &c. $s = \text{coſ. } 60^\circ = \tfrac{1}{2}$, $t = \text{coſ. } 180 = -1$, $u = 0$, &c. P = degrees in the arch whoſe ſine is $\frac{\sqrt{\frac{3}{4}}}{\sqrt{1 - \frac{z}{a} + \frac{zz}{aa}}}$, which is eaſily found from the Table of ſines in the ſame manner, as the degrees were found from the Table of Tangents in *Ex.* 12. Alſo Q = 0, &c. and by Caſe 1, $A = -1$, $B = \sqrt{3}$, $C = 1$, whence the Fluent is $\frac{1}{3aa} \times$ into $-$

$$2.30258 \text{ log. } \sqrt{1 - \frac{z}{a} + \frac{zz}{aa}} + .01745\sqrt{3} \times P$$

$$+ 2.30258 \times \text{log. } \overline{1 + \frac{z}{a}}.$$

EXAM. 31.

To find the Fluent of $\frac{dx^5\dot{x}}{a^8 - x^8}$.

Comparing this with Form 8, $z = x$, $\eta = 8$, $\alpha = a^8$, $\beta = -1$, $\frac{\delta}{\lambda} = \frac{3}{4}$, $K = 45$, $x = \frac{xx}{aa}$, alſo $a = s,\ 90 = 1$, $b = s,\ 180 = 0$, $c = 0$, &c. alſo we have $s = \text{coſ. } 90 = 0$, $t = \text{coſ. } 180 = 1$, $u = 0$, &c. P = degrees in the arch whoſe ſine is

$$\frac{xx}{aa\sqrt{1 + \frac{x^4}{a^4}}} \text{ or } \frac{xx}{\sqrt{a^4 + x^4}},$$ Q = 0, &c.

Alſo by Caſe 4, $A = -1$, $B = 0$, $C = -2$, $D = 1$.

All theſe being found, the Fluent is $\frac{d}{8aa} \times$ into

$$-\,2.30258 \text{ log. } \overline{1 - \frac{xx}{aa}} - 2 \times .017453\ P + 2.302585 \text{ log. } \overline{1 + \frac{xx}{aa}}.$$

Now ſuppoſe $a = 5$; $x = 4$, then $\frac{xx}{aa} = .64$, and $\text{log. } \overline{1 - \frac{xx}{aa}} = \text{log. } .36 = -\,1.556302 = -\,.443698$, and $\text{log. } \overline{1 + \frac{xx}{aa}} = \text{log. } 1.64 = 0.214844$.

Alſo $\frac{xx}{\sqrt{a^4 + x^4}} = .539053 = s.P$, therefore $P = 32.6192$ degrees. Whence the Fluent is $= 2.302585 \times .658542 - .017453 \times 65.2384 \times \frac{d}{200} = d \times .013275$; and this being wrought out at length, will ſerve as an example how to proceed in other ſuch like caſes.

EXAM. 32.

To find the Fluent of $\frac{\dot{x}}{\sqrt{aa + xx}}$.

Comparing this with Caſe 4, Form 9, we have $\alpha = aa$, $\beta = 1$, $z = x$, whence Fl. $\frac{\dot{x}}{\sqrt{aa + xx}} = 2.30258 \times \text{log. } x + \sqrt{aa + xx}$.

COR. *Alſo the Fluent of* $\frac{\dot{x}}{\sqrt{2ax + xx}}$ *is* $= 2.30258 \times \textit{log. } a + x + \sqrt{2ax + xx}$.

EXAM. 33.

To find the Fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{aa + 5x}}$.

Comparing this with Caſe 3, Form 9, we have $\alpha = aa$, $\beta = 5$, $z = x$, whence Fl. $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{aa+5x}} = \frac{2.30258}{\frac{1}{2}\sqrt{5}} \times$ log. $\sqrt{5x} + \sqrt{aa+5x}$.

EXAM. 34.

Let $\frac{2ax\dot{x}}{\sqrt{a^4+x^4}}$ *be propoſed.*

By Caſe 4, Form 9, we have $\alpha = a^4$, $\beta = 1$, $z^2 = x^4$, whence Fluent $= 2.30258 \times$ log. $\sqrt{x^4} + \sqrt{a^4+x^4}$. That is, the Fluent of $\frac{x\dot{x}}{\sqrt{a^4+x^4}} = 2.30258$ log. $xx + \sqrt{a^4+x^4}$. Therefore $2.30258 \times 2a$ log. $xx + \sqrt{a^4+x^4}$ is the Fluent required.

Or thus from the original Table.

By Form 9, $z = x$, $n = 4$, $\alpha = a^4$, $\beta = 1$, whence $\frac{2 \times 2.30258}{2} \times$ log. $\sqrt{x^4} + \sqrt{a^4+x^4}$, is the Fluent of $\frac{x\dot{x}}{\sqrt{a^4+x^4}}$, and conſequently $2.30258 \times 2a$ log. $xx + \sqrt{a^4+x^4}$ the Fluent required as before.

EXAM. 35.

To find the Fluent of $\frac{-4a\dot{x}}{\sqrt{2ax+xx}}$, *or of* $\frac{-4ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a+x}}$.

By Form 9, $z = x$, $n = 1$, $\alpha = 2a$, $\beta = 1$, whence Fluent $= 2.30258 \times 2$ log. $\sqrt{x} + \sqrt{2a+x}$; or rather, by the 2d way, Fluent $= 2.30258$ log. $2a + 2x + 2\sqrt{2ax+xx}$; or, by the 3d way, Fluent $= -2.30258$ log. $2a + 2x - 2\sqrt{2ax+xx}$.

Multiply by $-4a$, and we ſhall have the Fluent of $\frac{-4a\dot{x}}{\sqrt{2ax+xx}} = -2.3025 \times 8a$ log. $\sqrt{x} + \sqrt{2a+x}$

$\sqrt{2a+x}$: $= -2.30258 \times 4a$ log. $2a + 2x + 2\sqrt{2ax+xx} = 2.30258 \times 4a$ log. $2a + 2x - 2\sqrt{2ax+xx}$.

EXAM. 36.

To find the Fluent of $\frac{r\dot{y}}{y\sqrt{rr-yy}}$.

This quantity, as it ſtands, not agreeing with any of the forms, therefore divide both numerator and denominator by yy, then $\frac{r\dot{y}}{y\sqrt{rr-yy}} = \frac{ry^{-2}\dot{y}}{\sqrt{-1+rry^{-2}}}$, which belongs to Form 9.

Where $z = y$, $n = -2$, $\alpha = -1$, $\beta = rr$, whence $\varphi = \frac{-2 \times 2.302585}{2\sqrt{rr}} \times$ log. $rry^{-2} + \sqrt{-1+rry^{-2}}$: $= \frac{-2.3025}{r} \times$ log. $\frac{rr}{yy} + \sqrt{\frac{rr-yy}{yy}}$ and Fl. $\frac{r\dot{y}}{y\sqrt{rr-yy}} = -2.30258$ log. $\frac{rr}{yy} + \sqrt{\frac{rr-yy}{yy}}$.

EXAM. 37.

To find the Fluent of $\frac{\dot{x}\sqrt{x}}{\sqrt{a^2-7x^3}}$, *or* $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2-7x^3}}$.

Comparing this with Caſe 5, Form 10, we have $\alpha = aa$, $\beta = 7$, $z = x$, whence Fl. $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2-7x^3}} = \frac{.017453}{3\sqrt{7}} \times$ degrees in the arch, whoſe ſine $= \frac{2}{aa}\sqrt{7a^2x^3 - 49x^6}$.

EXAM. 38.

To find the Fluent of $\frac{-8a\dot{x}}{\sqrt{2ax-xx}}$, *or of* $\frac{-8ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}$.

Comparing this with Caſe 3, Form 10, we have $\alpha = 2a$, $\beta = 1$, $z = x$, whence Fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}$ $= .017453 \times$ deg. in the arch whoſe ſine is $= \frac{2}{2a}$ $\sqrt{2ax-xx}$; and Fl. $\frac{-8ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}} = -8a \times .017453$ $\times$ degrees in the arch whoſe ſine $= \frac{1}{a}\sqrt{2ax-xx}$.

EXAM. 39.

To find the Fluent of $\frac{2ax\dot{x}}{\sqrt{a^4-x^4}}$.

By reaſon of the negative quantity $-x^4$, compare this with Form 10, Caſe 6, where $\alpha = a^4$, $\beta = 1$, $z = x$, whence Fl. $\frac{x\dot{x}}{\sqrt{a^4-x^4}} = \frac{.017453}{2} \times$ degrees in the arch whoſe ſine is $\sqrt{\frac{x^4}{a^4}}$, or $\frac{xx}{aa}$; and the Fluent of $\frac{2ax\dot{x}}{\sqrt{a^4-x^4}}$ is $= .017453a \times$ degrees in the arch, whoſe ſine is $\frac{xx}{aa}$ and radius $= 1$.

EXAM. 40.

To find the Fluent of $\frac{-4a\dot{x}}{\sqrt{2ax-xx}}$, *or of* $\frac{-4ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}$.

By Caſe 3, Form 10, $\alpha = 2a$, $z = x$, $\beta = 1$, whence the Fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}} = .017453 \times$ degrees in the arch whoſe ſine is $\frac{2}{2a}\sqrt{2a-x \times x}$, and

Fl. of $\frac{-4ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}$, or of $\frac{-4a\dot{x}}{\sqrt{2ax-xx}}$ is $= -4a \times$ .017453 × degrees in the arch whoſe ſine is $\frac{1}{a}\sqrt{2ax-xx}$, and radius 1.

EXAM. 41.

To find the Fluent of $\frac{r\dot{y}}{y\sqrt{y^2-rr}}$.

Divide by yy, then $\frac{r\dot{y}}{y\sqrt{y^2-r^2}}$ is reduced to $\frac{ry^{-2}\dot{y}}{\sqrt{1-rry^{-2}}}$, which belongs to Form 10, Caſe 1, where $z=y$, $\alpha=1$, $\beta=rr$, ($n=-2$ in the original Table) whence Fl. $\frac{y^{-2}\dot{y}}{\sqrt{1-rry^{-2}}}$ or of $\frac{\dot{y}}{y\sqrt{yy-rr}}$ is $=\frac{.017453}{-\sqrt{rr}}$ × degrees in the arch whoſe ſine is $\sqrt{rry^{-2}}$ or $\sqrt{\frac{rr}{yy}}$. And Fluent of $\frac{r\dot{y}}{y\sqrt{yy-rr}}$ is $=$ − .017453 × degrees in the arch whoſe ſine is $\frac{r}{y}$, rad. $= 1$.

HAVING gone through the firſt ten Forms, we ſhall now proceed to the 11th, 12th, 13th, and 14th. Before you can find the Fluent of a Fluxion belonging to any of theſe Forms, you muſt firſt have the Fluent of $\overline{\alpha+\beta z^{n}}|^{\mu}z^{\pi}\dot{z}$, or at leaſt know that you can find it. For this is the key to all the reſt, without which it will be needleſs to proceed any further.

Therefore when any Fluxion is propoſed, obſerve well what your n is; and if by continually ſubtracting or adding n to the index of z, you come at laſt to $\eta - 1$, or $\frac{1}{2}\eta - 1$, for the value of π; then ſome of the foregoing Forms will give you the Fluent of $\overline{\alpha+\beta z^{\eta}}|^{\mu} z^{\pi}\dot{z} = \phi$, and from this, the Fluent required is had by Form 11 or 12.

Likewiſe by continually ſubtracting or adding 1 to the index of the radical, if you come at laſt to $-\frac{1}{2}$ for the value of μ, then ſome of the foregoing Forms will give the Fluent of $\overline{\alpha+\beta z^{\eta}}|^{\mu} z^{\pi}\dot{z}$, and the 13th or 14th Form, that of the Fluxion propoſed.

But to ſave ſome labour, conſult the notes in the margin of theſe Forms, by which you will know whether the Fluent can be had by that Form or not. If not, then your buſineſs is to transform your Fluent to another Form (by dividing it by the variable quantity under the *vinculum*). And then the ſame proceſs is to be repeated as before; and if neither way will do, then the Caſe is deſperate.

From what has been ſaid it is plain, that if the Fl. $\overline{\alpha+\beta z^{\eta}}|^{\mu} z^{\pi}\dot{z}$ be but known, the Fluents of $\overline{\alpha+\beta z^{\eta}}|^{\mu}$ × into $z^{\pi+\eta}\dot{z}$, $z^{\pi+2\eta}\dot{z}$, $z^{\pi+3\eta}\dot{z}$, &c. or in general $z^{\pi+\lambda\eta}\dot{z}$ will be known by Form 11; and the Fluents of $\overline{\alpha+\beta z^{\eta}}|^{\mu}$ × into $z^{\pi-\eta}\dot{z}$, $z^{\pi-2\eta}\dot{z}$, $z^{\pi-3\eta}\dot{z}$, &c. or in general $z^{\pi-\lambda\eta}\dot{z}$, by Form 12. And alſo the Fluents of $\overline{\alpha+\beta z^{\eta}}|^{\mu\pm 1} z^{\pi}\dot{z}$, $\overline{\alpha+\beta z^{\eta}}|^{\mu\pm 2} z^{\pi}\dot{z}$, &c. by Form 13 and 14: except the caſes mentioned in the margin. This being laid down in general, proceed we to ſome examples.

EXAM. 42.

To find the Fluent of $\dfrac{x^4\dot{x}}{\sqrt{rr+xx}}$.

Here I ſee plainly, that if I had the Fluent of $\dfrac{\dot{x}}{\sqrt{rr+xx}}$, I could ſoon find the Fluent of $\dfrac{x^2\dot{x}}{\sqrt{rr+xx}}$,

$\frac{x^4\dot{x}}{\sqrt{rr+xx}}$, $\frac{x^6\dot{x}}{\sqrt{rr+xx}}$, and ſo on by Form 11. Now the Fl. $\frac{\dot{x}}{\sqrt{rr+xx}}$ or $\frac{x^{1-1}\dot{x}}{\sqrt{rr+xx}}$ is had by Form 9, and is $2.30258 \times \log. x+\sqrt{r^2+x^2}=\varphi$. Whence by Form 11, we have $\alpha=rr$, $\beta=1$, $z=x$, $\eta=2$, $\pi=0$, $\mu=-\frac{1}{2}$, $\lambda=2$, $\gamma=\sqrt{rr+xx}$, $\varepsilon=(\pi+1+\lambda\eta)\,5$, $\delta=(\varepsilon+\eta+\mu\eta)\,6$. Whence the Fluent is $=\frac{\overline{-rr}|^{2}.3.1}{1.4.2}\varphi+\frac{x^3}{4}\gamma-\frac{3rr}{2xx}\mathrm{A}=\frac{3r^4}{8}\varphi+\frac{x^3}{4}\sqrt{rr+xx}-\frac{3rrx}{8}\sqrt{rr+xx}$; that is,

$$\text{Fl. }\frac{x^4\dot{x}}{\sqrt{rr+xx}}=\frac{3r^4}{8}\times 2.30258\log. x+\sqrt{rr+xx}$$
$$+\frac{2x^3-3r^2x}{8}\sqrt{rr+xx}.$$

EXAM. 43.

To find the Fluent of $\frac{d\dot{y}}{yy\sqrt{rr-yy}}$.

This I can find by Form 12, if I can but get the Fluent of $\frac{\dot{y}}{\sqrt{rr-yy}}$. But by Form 10, Fl. $\frac{\dot{y}}{\sqrt{rr-yy}}=.01745\times$ degrees in the arch whoſe ſine is $\frac{y}{r}$, rad. $=1$. Therefore by Form 12, $\alpha=rr$, $\beta=-1$, $z=y$, $\eta=2$, $\mu=-\frac{1}{2}$, $\pi=0$, $\lambda=1$, $\gamma=\sqrt{rr-yy}$, $\varepsilon=-3$, $\delta=-2$, and put Fl. $\frac{y}{\sqrt{rr-yy}}=\varphi$, then Fl. $\frac{\dot{y}}{yy\sqrt{rr-yy}}$ or

$y^{-2}\dot{y}$

$\frac{y^{-2}\dot{y}}{\sqrt{rr-yy}} = \frac{-1.0}{-rr.\times-1}\varphi + \frac{y^{-1}\gamma}{-rr} = 0 - \frac{\gamma}{rry} =$ $-\frac{\sqrt{rr-yy}}{rry}$, and Fl. $\frac{d\dot{y}}{yy\sqrt{rr-yy}} = \frac{-d\sqrt{rr-yy}}{rry}$.

EXAM. 44.

To find the Fluent of $x^5\dot{x}\sqrt[3]{a^3-b^3+\frac{a}{b}x^3}$.

Here the Fluent of $x^2\dot{x}\sqrt[3]{a^3-b^3+\frac{a}{b}x^3}$ (by Form 3,) $= \frac{b.\overline{a^3-b^3+\frac{a}{b}x^3}|^{\frac{4}{3}}}{4a} = \varphi$; then by Form 11, $\alpha = a^3-b^3$, $\beta = \frac{a}{b}$, $z = x$, $n = 3$, $\mu = \frac{1}{3}$, $\pi = 2$, $\lambda = 1$, $\varepsilon = 6$, $\delta = 10$, $\gamma = \overline{a^3-b^3+\frac{a}{b}x^3}|^{\frac{4}{3}}$, whence Fl. $x^5\dot{x}\sqrt[3]{a^3-b^3+\frac{a}{b}x^3} = \frac{\overline{b^3-a^3}\times b.3}{a.7}$ $\varphi + \frac{x^3\gamma b}{7a} = \frac{3.\overline{b^3-a^3}}{7a}b\varphi + \frac{bx^3}{7a}\times\overline{a^3-b^3+\frac{a}{b}x^3}|^{\frac{4}{3}}$.

EXAM. 45.

To find the Fluent of $\frac{d\dot{v}}{v^4\sqrt{av-vv}}$.

This reduced is $\frac{dv^{-4\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$. Now if I can find the Fl. $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, then by Form 12, I can find the Fl. $\frac{v^{-1\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, $\frac{v^{-2\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, $\frac{v^{-3\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, $\frac{v^{-4\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, and ſo on. But by Form 10, Fl. $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a-v}} = .01745 \times 2$ degrees in the arch whoſe ſine is $\sqrt{\frac{v}{a}}$, rad. $= 1$. Let this Fluent $= \varphi$, then by Form 12, $\alpha = a$, $\beta = -1$, $z = v$, $n = 1$, $\mu = -\frac{1}{2}$, $\lambda = 4$, $\gamma = \sqrt{a-v}$, $\epsilon = -4\frac{1}{2}$, $\delta = -4$. Whence

$$\text{Fl. } \frac{v^{-4\frac{1}{2}}\dot{v}}{\sqrt{a-v}} = \frac{\overline{-1}|^{4}. -3. -2. -1.0}{\overline{-a}|^{4}. -3\frac{1}{2}. -2\frac{1}{2}. -1\frac{1}{2}. -\frac{1}{2}}\,\varphi +$$

$$\frac{v^{-3\frac{1}{2}}\gamma}{-3\frac{1}{2}a} + \frac{3v}{2\frac{1}{2}a}A + \frac{2v}{1\frac{1}{2}a}B + \frac{v}{\frac{1}{2}a}C = -\frac{2\gamma}{7v^{3\frac{1}{2}}a}$$

$$+ \frac{6v}{5a}A + \frac{4v}{3a}B + \frac{2v}{a}C.$$ And

$$\text{Fl. } \frac{dv^{-4\frac{1}{2}}\dot{v}}{\sqrt{a-v}} = \frac{-2d\sqrt{av-vv}}{7v^{4}a} + \frac{6v}{5a}A + \frac{4v}{3a}B + \frac{2v}{a}C.$$

EXAM. 46.

To find the Fluent of $\frac{c\dot{x}}{ds}\sqrt{\frac{ddss}{cc} + xx}$.

Here if I had the Fluent of $\frac{\dot{x}}{\sqrt{\frac{ddss}{cc} + xx}}$, I could then find the Fluent of $\dot{x}\sqrt{\frac{ddss}{cc} + xx}$, or that of $\frac{c\dot{x}}{ds}\sqrt{\frac{ddss}{cc} + xx}$, by Form 13. But the Fl. of $\frac{\dot{x}}{\sqrt{\frac{ddss}{cc} + xx}}$, is had by Form 9, and is

$$2.30258 \times \log.\ x + \sqrt{\frac{ddss}{cc} + xx} := \varphi.$$

Then by Form 13, $\alpha = \frac{ddss}{cc}$, $\beta = 1$, $z = x$, $\eta = 2$, $\pi = 0$, $\mu = -\frac{1}{2}$, $\tau = 1$, $\gamma = \frac{ddss}{cc} + xx$, $\vartheta = (\pi + 1 + \mu\eta)\, 0$, and Fl. $\dot{x}\sqrt{\frac{ddss}{cc} + xx} =$

$$\frac{\frac{2ddss}{cc} \times \frac{1}{2}}{2}\varphi + \frac{x\gamma^{\frac{1}{2}}}{2} = \frac{ddss}{2cc}\varphi + \frac{x}{2}\sqrt{\frac{ddss}{cc} + xx}.$$

And Fl. $\frac{c\dot{x}}{ds}\sqrt{\frac{ddss}{cc} + xx} = \frac{ds}{2c}\varphi + \frac{cx}{2ds}\sqrt{\frac{ddss}{cc} + xx}$ $= \frac{ds}{2c} \times 2.3025 \times \log.\ x + \sqrt{\frac{ddss}{cc} + xx} : +$ $\frac{2x}{2ds}\sqrt{\frac{ddss}{cc} + xx}$.

EXAM. 47.

To find the Fluent of $-2f\sqrt{\frac{1}{4}pp + gyy} \times .7854\dot{y}$, *which is in Question* 210, *Ladies Diary.*

First I find the Fluent of $\frac{\dot{y}}{\sqrt{\frac{1}{4}pp + gyy}}$ by Form 9, which is $\frac{2.30258}{\sqrt{g}} \times \log.\ \sqrt{gyy} + \sqrt{\frac{1}{4}pp + gyy} :$ $= \varphi$.

Then by Form 13, $\alpha = \frac{1}{4}pp$, $\beta = g$, $z = y$, $\eta = 2$, $\pi = 0$, $\mu = -\frac{1}{2}$, $\tau = 1$, $\gamma = \sqrt{\frac{1}{4}pp + gyy} =$ $\frac{\frac{2}{4}pp \times \frac{1}{2}}{2}\varphi + \frac{y\gamma^{\frac{1}{2}}}{2} = \frac{pp}{8}\varphi + \frac{y}{2}\sqrt{\gamma}$, and Fl. $-2f\sqrt{\frac{1}{4}pp + gyy} \times .7854\dot{y} = \frac{-fpp}{4} \times .7854\varphi - fy$ $\times .7854\sqrt{\frac{1}{4}pp + gyy}$.

EXAM.

EXAM. 48.

To find the Fluent of $\frac{4\dot{x}}{3a} \times \overline{ee-xx}|^{\frac{3}{2}}$.

By Form 10, the Fluent of $\frac{\dot{x}}{\sqrt{ee-xx}}$ is $=.01745$ $\times$ degrees in the arch whoſe ſine is $\frac{x}{e}$, and rad. $=$ 1; that is, the Fl. of $\frac{\dot{x}}{\sqrt{ee-xx}} = \varphi$. And comparing $\dot{x} \times \overline{ee-xx}|^{\frac{3}{2}}$ with Form 13, we have $\eta=2$, $\pi = v$, $\tau = 2$, $\mu = -\frac{1}{2}$, $\alpha = ee$, $\beta = -1$, $\delta=0$, whence Fl. $\dot{x} \times \overline{ee-xx}|^{\frac{3}{2}} = \frac{4.e^4.\frac{1}{2}.\frac{3}{2}}{2.4}\varphi + \frac{x \times \overline{ee-xx}|^{\frac{3}{2}}}{4}$ $+ \frac{\frac{3}{2}.2ee}{2.\overline{ee-xx}}$ A; that is, $\frac{3e^4}{8}\varphi + \frac{1}{4}x.\overline{ee-xx}|^{\frac{3}{2}} +$ $\frac{3eex}{2}\sqrt{ee-xx}$, or,

$\frac{3e^4}{8}\varphi + \frac{7eex-x^3}{4}\sqrt{ee-xx}$, this multiplied by $\frac{4}{3a}$, gives the Fluent required.

EXAM. 49.

To find the Fluent of $\frac{d\dot{z}}{\overline{aa-zz}|^{2}}$.

Firſt by Form 6, I find the Fluent of $\frac{\dot{z}}{aa-zz}$ $= \frac{2.3025}{2a} \times \log. \frac{a+z}{a-z} = \varphi$. Then by Form 14, $\alpha = aa$, $\beta = -1$, $\eta=2$, $\pi = 0$, $\mu = -1$, $\tau=1$, $\gamma = aa-zz$, $\delta = (\pi + 1 + \mu\eta =) - 1$. Then Fl. $\frac{\dot{z}}{\overline{aa-zz}|^{2}} = \frac{-1}{2aa \times -1}\varphi - \frac{z}{-1 \times 2aa \times \overline{aa-zz}}$.

$=$

$= \frac{\varphi}{2aa} + \frac{z}{2aa \times \overline{aa - zz}}$, and Fl. $\frac{d\dot{z}}{\overline{aa - zz}|^{2}} = \frac{d\varphi}{2aa}$
$+ \frac{d\dot{z}}{2aa \times \overline{aa - zz}}$.

EXAM. 50.

To find the Fluent of $\frac{2cy\dot{y}}{\overline{aa + \frac{b}{c}yy}|^{\frac{5}{2}}}$, *or of* $2cy\dot{y} \times \overline{aa + \frac{b}{c}yy}|^{-\frac{5}{2}}$.

By Form 3, Fl. $\frac{y\dot{y}}{\sqrt{aa + \frac{b}{c}yy}}$, or $y\dot{y} \times \overline{aa + \frac{b}{c}yy}|^{-\frac{1}{2}}$

$= \frac{\sqrt{aa + \frac{b}{c}yy}}{\frac{1}{2} \times 2\frac{b}{c}} = \frac{c}{b}\sqrt{aa + \frac{b}{c}yy} = \varphi$. And by Form 14, $\alpha = aa$, $\beta = \frac{b}{c}$, $n = 2$, $\pi = 1$, $\mu = -\frac{1}{2}$, $\tau = 2$, $\gamma = aa + \frac{b}{c}yy$, $z = y$, $\delta =$ $(\pi + 1 + \mu n =)$ 1, then

Fluent of $\frac{y\dot{y}}{\overline{aa + \frac{b}{c}yy}|^{\frac{5}{2}}} = \frac{1 \times -1}{4a^4 \times -\frac{1}{2} \times -\frac{3}{2}}\varphi - \frac{yy\gamma^{-1\frac{1}{2}}}{-1\frac{1}{2} \times 2aa} + \frac{-\gamma}{-\frac{1}{2} \times 2aa}A = \frac{-\varphi}{3a^4} + \frac{yy}{3aa \times \overline{aa + \frac{b}{c}yy}|^{\frac{3}{2}}} + \frac{yy}{3a^4 \times \overline{aa + \frac{b}{c}yy}|^{\frac{1}{2}}}$. And Fl.

2cyẏ

$$\frac{2cy\dot{y}}{\overline{aa+\frac{b}{c}yy}\big|^{\frac{5}{2}}} = \frac{-2c\varphi}{3a^4} + \frac{2cyy}{3aa\times\overline{aa+\frac{b}{c}yy}\big|^{\frac{3}{2}}} +$$

$$\frac{2cyy}{3a^4\times\overline{aa+\frac{b}{c}yy}\big|^{\frac{1}{2}}}.$$

EXAM. 51.

To find the Fluent of $\frac{cx^4\dot{x}}{ds}\sqrt{\frac{adss}{cc}+xx}$.

Proceeding as in *Example* 46, Fl. $\frac{c\dot{x}}{ds}\sqrt{\frac{ddss}{cc}+xx}$ $= \frac{ds}{2c}\times 2.30258\times \text{log. } x+\sqrt{\frac{ddss}{cc}+xx}:+$ $\frac{2x}{2ds}\sqrt{\frac{ddss}{cc}+xx}=\varphi$. Then by Form 11, $\alpha = \frac{ddss}{cc}$, $\beta=1$, $z=x$, $\eta=2$, $\pi=0$, $\mu=\frac{1}{2}$, $\lambda=2$, $\gamma=\overline{\frac{ddss}{cc}+xx}\big|^{\frac{3}{2}}$, $\varepsilon=5$, $\delta=8$; whence Fl. $\frac{cx^4\dot{x}}{ds}$ $\sqrt{\frac{ddss}{cc}+xx} = \frac{d^4s^4.3.1}{c^4.6.4}\varphi+\frac{x^3\gamma}{6}-\frac{3ddss}{cc.4xx}\text{A} =$ $\frac{d^4s^4}{8c^4}\varphi+\frac{x^3}{6}\times\overline{\frac{ddss}{cc}+xx}\big|^{\frac{3}{2}}-\frac{3ddss}{4ccxx}\times\frac{x^3}{6}\times$ $\overline{\frac{ddss}{cc}+xx}\big|^{\frac{3}{2}} = \frac{d^4s^4}{8c^4}\varphi+\frac{x^3}{6}-\frac{3ddss}{4ccxx}\times\overline{\frac{ddss}{cc}+xx}\big|^{\frac{3}{2}}$.

EXAM. 52.

To find the Fluent of $\frac{\frac{1}{2}av\dot{v}}{1+v^4}-\frac{2av^5\dot{v}}{\overline{1+v^4}\big|^2}$.

1. Given the Fluent of $\frac{v\dot{v}}{1+v^4}=\frac{.017453}{2}\times$ de-

grees in the arch whoſe tangent is $vv = \varphi$, by Form 5.

To find the Fluent of $\frac{v^5\dot{v}}{1+v^4}$.

By Form 11, $n = 4$, $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\mu = -1$, $\pi = 1$, $\varepsilon = 6$, $\delta = 6$, $\gamma = 1$; then the Fl. $\frac{v^5\dot{v}}{1+v^4} = \frac{-1\times 2}{1\times 2}\varphi + \frac{\dot{v}^2.1}{2.1} = \frac{1}{2}vv - \varphi$.

2. Given the Fl. $\frac{v^5\dot{v}}{1+v^4} = \varphi$, to find that of $\frac{v^5\dot{v}}{\overline{1+v^4}|^2}$.

By Form 14, $\pi = 5$, $-\tau = -1$, $\delta = 2$, $\gamma = 1+v^4$, the reſt as before; then Fl. $\frac{v^5\dot{v}}{\overline{1+v^4}|^2} = \frac{2}{4.1.-1}\varphi - \frac{v^6\times\overline{1+v^4}|^{1-1}}{-1\times 4.1} = \frac{v^6}{4\times\overline{1+v^4}} - \frac{1}{2}\varphi$.

Therefore Fl. $\frac{2av^5\dot{v}}{\overline{1+v^4}|^2} = \frac{av^6}{2\times\overline{1+v^4}} - a\varphi = \frac{av^6}{2\times\overline{1+v^4}} - a\times\overline{\frac{1}{2}vv - \varphi} = \frac{\frac{1}{2}av^6}{1+v^4} - \frac{1}{2}avv + a\varphi$.

Therefore Fl. $\frac{\frac{1}{2}av\dot{v}}{1+v^4} - \frac{2av^5\dot{v}}{\overline{1+v^4}|^2} = \frac{1}{2}av - \frac{1}{2}a\varphi - \frac{\frac{1}{2}av^6}{1+v^4}$.

But $\frac{1}{2}av^2 = \frac{\frac{1}{2}av^2\times\overline{1+v^4}}{1+v^4} = \frac{\frac{1}{2}av^2 + \frac{1}{2}av^6}{1+v^4}$, therefore $\frac{1}{2}av^2 - \frac{1}{2}a\varphi - \frac{\frac{1}{2}av^6}{1+v^4} = -\frac{1}{2}a\varphi + \frac{\frac{1}{2}av^2 + \frac{1}{2}av^6 - \frac{1}{2}av^6}{1+v^4} = -\frac{1}{2}a\varphi + \frac{\frac{1}{2}avv}{1+v^4}$, the Fluent required.

EXAM. 53.

To find the Fluent of $\dot{x} \times \overline{\sqrt{x} + \sqrt{a+x}}|^{n}$.

Let $y^2 = x$, and $v^2 = a + x$, then $\dot{x} = 2y\dot{y}$, and we ſhall have $2y\dot{y} \times \overline{y+v}|^{n}$ for the fluxionary quantity; this by involving becomes $2y\dot{y} \times y^{n} + ny^{n-1}v + \frac{n.\overline{n-1}}{2} y^{n-2}v^{2}$ &c. = (by reſtoring the value of v) $2y^{n+1}\dot{y} + 2ny^{n}\dot{y}\sqrt{a+y^{2}} + \frac{2n.\overline{n-1}}{2} y^{n-1}\dot{y} \times \overline{a+yy} + 2n. \frac{\overline{n-1}}{2} . \frac{n-2}{3} y^{n-2}\dot{y} \times \overline{a+yy}|^{\frac{3}{2}}$, &c.

Here the 1ſt, 3d, 5th, terms will conſiſt of ſimple terms only, whoſe Fluents are eaſily found. To find the Fluents of the 2d, 4th, 6th terms, proceed thus.

CASE I. Let n be an odd number, then will $n-1$, $n-3$, $n-5$, &c. be even numbers, and it is evident, that the Fluent of $\frac{y\dot{y}}{\sqrt{a+yy}}$ is $\sqrt{a+yy}$, which call φ, then it is plain that φ being given the Fluents of $\frac{y^{n}\dot{y}}{\sqrt{a+yy}}$, $\frac{y^{n-2}\dot{y}}{\sqrt{a+yy}}$ &c. will be given by Form 11, and when theſe are had, the Fluents of $y^{n}\dot{y}\sqrt{a+yy}$, $y^{n-2}\dot{y} \times \overline{a+yy}|^{\frac{3}{2}}$ &c. will be had by *Form* 13. Thereiore we ſhall have the Fluent of all the terms, when n is odd.

CASE II. Let n be even, then will n, $n-2$, $n-4$, &c. be even; and it is evident, that the Fluent of $\frac{\dot{y}}{\sqrt{a+yy}}$ by *Form* 9 is had, which call φ, then φ being had the Fluents of $\frac{y^{n}\dot{y}}{\sqrt{a+yy}}$, $\frac{y^{n-2}\dot{y}}{\sqrt{a+yy}}$ &c. will be had by *Form* 11, as before, and thence the Fluents of $y^{n}\dot{y}\sqrt{a+yy}$, $y^{n-2}\dot{y} \times \overline{a+yy}|^{\frac{3}{2}}$ &c. by *Form*

13,

13, and when *n* is aſſigned in numbers, the particular Fluents will be obtained as above.

Thus have I here given a great variety of examples for finding Fluents by the *firſt fourteen Forms* of that moſt excellent Table in Mr. *Emerſon*'s Fluxions; what follows after will be about infinite ſeries, and tranſmutation of Fluxions, the purſuing of their explanations, will in no wiſe ſuit my preſent deſign, and therefore muſt refer the reader to the book itſelf.

As the finding of Fluents is a moſt difficult Problem, and yet of the greateſt uſe in the whole practice of Fluxions, ſo the learned Author has calculated his Table on purpoſe with a deſign to remedy this difficulty; its uſe will appear upon trial to be far more eaſy and intelligible than any extant, and no leſs general; for when the Table is to be made uſe of, there is not the leaſt difficulty in it, there being nothing required but a bare ſubſtitution of quantities, which I now think to be the eaſieſt way in the world; and I can hardly underſtand how any thing can be contrived more eaſy and at the ſame time as general. Since therefore, as Fluents for the moſt part, however in abundance of inſtances, are much ſooner found by this Table than any other way, I much wonder it is not made more uſe of, ſeeing it is ſo eaſy, and the Fluents come out ſo neat and ſimple; yet many love to have recourſe to infinite ſeries, which is both very laborious and troubleſome, though at the ſame time the Fluent by the Table may be obtained in finite terms. But if people will be prejudiced, and ſuffer themſelves to wade in intricate depths, when they might go on eaſily and chearfully in the open plains, I cannot help it, but muſt leave them to chuſe their own ways. Yet leaſt I ſhould be thought to depreciate the method of finding Fluents by infinite ſeries, it is far from me; for I am well aſſured, that in many inſtances, Fluents cannot be obtained in any other way but by infinite ſeries, and often with much difficulty and labour in that way too; but I would here be underſtood to mean, when a Fluent can be had by the aforeſaid Table, it is by far the more preferable way, and which I recommend my Readers to be ready at, for which purpoſe the foregoing Examples are given, to aſſiſt them herein.

I ſhall now, according to my purpoſe, give a few Examples for finding Fluents by an infinite ſeries.

EXAM. 1.

Required the Fluent of $\frac{\dot{x}}{1+xx}$, *or of* $\dot{x} \times \frac{1}{1+xx}$.

RULE. 7.

Put the Fluxion into a ſeries by diviſion, &c.

Thus 1. $\frac{1}{1+xx} = 1 - x^2 + x^4 - x^6 + x^8$ &c. by *Ex.* 3. *Sect.* II.

2. Which multiplied by $\dot{x}$ produces $\dot{x} - x^2\dot{x} + x^4\dot{x} - x^6\dot{x} + x^8\dot{x}$ &c. then by *Rule* 2.

The Fluent of $\dot{x}$ is $= x$

of $-\,x^2\dot{x} = -\frac{x^3}{3}$

of $+\,x^4\dot{x} = +\frac{x^5}{5}$

of $-\,x^6\dot{x} = -\frac{x^7}{7}$

of $+\,x^8\dot{x} = +\frac{x^9}{9}$

&c.

Therefore Fluent is $= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}$ &c. in an infinite ſeries.

Now the Fluent of $\frac{\dot{x}}{1+xx}$ is $= 2.30258 \times$ log. $\overline{1+xx}$, by the Table, and is had by inſpection.

EXAM. 2.

To find the Fluent of $\dot{x}\sqrt{aa+xx}$ *in an infinite ſeries?*

1. The ſquare root of $aa+xx = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c.

2. Multiply the ſeries by $\dot{x}$ is $= a\dot{x} + \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} + \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$, &c.

3. Find the Fluent of every quantity ſeparately by *Rule* 2, and the Fluent is $= ax + \frac{x^3}{3.2a} - \frac{x^5}{5.8a^3} + \frac{x^7}{7.16a^5} - \frac{5x^9}{9.128a^7}$ &c. or $= a\dot{x} + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. the Fluent required. Or by the Table it is $\frac{2}{3} aa + xx \sqrt{aa+xx}$ by Form 3.

EXAM. 3.

To find the Fluent of $2\dot{x}\sqrt{aa+xx}$ *in an infinite ſeries.*

1. $\sqrt{aa+xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. by Sect. II.

2. Multiply by $2\dot{x}$ is $= 2a\dot{x} + \frac{x^2\dot{x}}{a} - \frac{x^4\dot{x}}{4a^3} + \frac{x^6\dot{x}}{8a^5} - \frac{5x^8\dot{x}}{64a^7}$ &c. whoſe

3. Fluent $= 2ax + \frac{x^3}{3a} - \frac{x^5}{5.4a^3} + \frac{x^7}{7.8a^5} - \frac{5x^9}{9.64a^7}$ &c. by Rule 2.

That is, Fl. $2\dot{x}\sqrt{aa+xx} = 2ax + \frac{x^3}{3a} - \frac{x^5}{20a^3} + \frac{x^7}{56a^5} - \frac{5x^9}{576a^7}$ &c.

EXAM. 4.

Required the Fluent of $\frac{\dot{x}}{aa}\sqrt{a^4 - b^2x^2}$?

1. $\sqrt{a^4 - b^2x^2} = a^2 - \frac{b^2x^2}{2aa} - \frac{b^4x^4}{8a^6} - \frac{b^6x^6}{16a^{10}} - \frac{5b^8x^8}{128a^{14}}$ &c.

2. Mult. by $\frac{\dot{x}}{aa}$ produces $\dot{x} - \frac{b^2x^2\dot{x}}{2a^4} - \frac{b^4x^4\dot{x}}{8a^8} - \frac{b^6x^6\dot{x}}{16a^{12}} - \frac{5b^8x^8\dot{x}}{128a^{16}}$ &c.

3. Fluent $= x - \frac{b^2x^3}{3.2a^4} - \frac{b^4x^5}{5.8a^8} - \frac{b^6x^7}{7.16a^{12}} - \frac{5b^8x^9}{9.128a^{10}}$ &c.

EXAM. 5.

To find the Fluent of $x^{n+1}\dot{x} - \frac{1}{3}x^{n+3}\dot{x} + \frac{1}{5}x^{n+5}\dot{x} - \frac{1}{7}x^{n+7}\dot{x}$ *&c.*

By Rule 2, Fluent of $x^{n+1}\dot{x} = \frac{x^{n+2}}{n+2}$.

——— of $\frac{1}{3}x^{n+3}\dot{x} = \frac{x^{n+4}}{3.\overline{n+4}}$.

By Rule 2, Fluent of $\frac{1}{5}x^{n+5}\dot{x} = \frac{x^{n+6}}{5\,\overline{n+6}}$.

—— of $\frac{1}{7}x^{n+7}\dot{x} = \frac{x^{n+8}}{7.\overline{n+8}}$.

&c.

Therefore the Fluent is $= \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3.\overline{n+4}} + \frac{x^{n+6}}{5.\overline{n+6}} - \frac{x^{n+8}}{7.\overline{n+8}}$ &c.

EXAM. 6.

To find the Fluent of $\dot{x} \times \overline{aa-xx}|^{\frac{3}{4}}$?

1. $\overline{aa-xx}|^{\frac{3}{4}} = a^{\frac{3}{2}} - \frac{3x^2}{4a^{\frac{1}{2}}} - \frac{3x^4}{4.8a^{\frac{5}{2}}} - \frac{3.5x^6}{4.8.12a^{\frac{9}{2}}} - \frac{3.5.9x^8}{4.8.12.16a^{\frac{13}{2}}}$ &c.

2. Mult. by $\dot{x} = a^{\frac{3}{2}}\dot{x} - \frac{3x^2\dot{x}}{4a^{\frac{1}{2}}} - \frac{3x^4\dot{x}}{4.8a^{\frac{5}{2}}} - \frac{3.5x^6\dot{x}}{4.8.12a^{\frac{9}{2}}} - \frac{3.5.9x^8\dot{x}}{4.8.12.16a^{\frac{13}{2}}}$ &c.

3. Fluent of $a^{\frac{3}{2}}\dot{x} = a^{\frac{3}{2}}x$

—— of $\frac{3x^2\dot{x}}{4a^{\frac{1}{2}}} = \frac{3.x^3}{3.4a^{\frac{1}{2}}} = \frac{x^3}{4a^{\frac{1}{2}}}$

—— of $\frac{3x^4\dot{x}}{4.8a^{\frac{5}{2}}} = \frac{3x^5}{5.4.8a^{\frac{5}{2}}}$

—— of $\frac{3.5x^6\dot{x}}{4.8.12a^{\frac{9}{2}}} = \frac{3.5x^7}{7.4.8.12a^{\frac{9}{2}}}$

—— of $\frac{3.5.9x^8\dot{x}}{4.8.12.16a^{\frac{13}{2}}} = \frac{3.5.9x^9}{9.4.8.12.16a^{\frac{13}{2}}}$

Therefore Fluent $= a^{\frac{3}{2}}x - \frac{x^3}{4a^{\frac{1}{2}}} - \frac{3x^5}{5.4.8a^{\frac{5}{2}}} - \frac{3.5x^7}{7.4.8.12a^{\frac{9}{2}}} - \frac{3.5.9x^9}{9.4.8.12.16a^{\frac{13}{2}}}$ &c.

Ex.

EXAM. 7.

To find the Fluent of $\dfrac{\dfrac{-\dot{x}}{2\sqrt{x-xx}}}{\sqrt{a-x}}$, *or of* $\dfrac{-\dot{x}}{2\sqrt{x-xx}}$ $\times\sqrt{a-x}$.

1. $\sqrt{x-xx} = x^{\frac{1}{2}} - \dfrac{x^{\frac{3}{2}}}{2} - \dfrac{x^{\frac{5}{2}}}{8} - \dfrac{x^{\frac{7}{2}}}{116} - \dfrac{5x^{\frac{9}{2}}}{128}$ &c. by *Sect.* II.

2. $x^{\frac{1}{2}} - \dfrac{x^{\frac{3}{2}}}{2} - \dfrac{x^{\frac{5}{2}}}{8}$ &c. $\Big)\, 1 \,\Big(\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{x^{\frac{1}{2}}}{2} + \dfrac{3x^{\frac{3}{2}}}{2.4}$ &c. by diviſion.

3. $\dfrac{-\dot{x}}{2\sqrt{x-xx}} = \dfrac{1}{2} \times - \dfrac{\dot{x}}{x^{\frac{1}{2}}} - \dfrac{x^{\frac{1}{2}}\dot{x}}{2} - \dfrac{3x^{\frac{3}{2}}\dot{x}}{2.4}$ &c.

4. $\dfrac{1}{\sqrt{a-x}} = \dfrac{1}{a^{\frac{1}{2}}} + \dfrac{x}{2a^{\frac{3}{2}}} + \dfrac{3x^2}{2.4a^{\frac{5}{2}}}$ &c. after the ſquare root of $a-x$ is extracted by Sect. II and 1 divided by that root as above, or it is $= \dfrac{1}{a^{\frac{1}{2}}} \times 1 + \dfrac{x}{2a} + \dfrac{3x^2}{2.4a^2}$ &c. which two ſeries multiplied together, viz. $\dfrac{-\dot{x}}{2\sqrt{x-xx}} \times \sqrt{a-x} = \dfrac{1}{2a^{\frac{1}{2}}} \times - \dfrac{\dot{x}}{x^{\frac{1}{2}}}$

$$- \frac{x^{\frac{1}{2}}\dot{x}}{2} - \frac{3x^{\frac{3}{2}}\dot{x}}{2.4} \text{ \&c.}$$

$$- \frac{x^{\frac{1}{2}}\dot{x}}{2a} - \frac{x^{\frac{3}{2}}\dot{x}}{2.2a} - \frac{3x^{\frac{5}{2}}\dot{x}}{2.4.2a} \text{ \&c.}$$

$$- \frac{3x^{\frac{3}{2}}\dot{x}}{2.4a^2} - \frac{3x^{\frac{5}{2}}\dot{x}}{2.4.2a^2} - \frac{3.3x^{\frac{7}{2}}\dot{x}}{2.4.2.4a^2} \text{ \&c.}$$

Fluent is $= \dfrac{1}{2a^{\frac{1}{2}}} \times - 2x^{\frac{1}{2}} - \dfrac{2x^{\frac{3}{2}}}{2.3} - \dfrac{2.3x^{\frac{5}{2}}}{2.4.5}$ &c.

$- 2x^{\frac{3}{2}}$

$$-\frac{2x^{\frac{3}{2}}}{2.3a} - \frac{2x^{\frac{5}{2}}}{2.2.5a} - \frac{2.3x^{\frac{7}{2}}}{2.4.2.7a} \text{ \&c.}$$

$$-\frac{2\ 3x^{\frac{5}{2}}}{2.4.5a^2} - \frac{2.3x^{\frac{7}{2}}}{2.2.4.7a^2} - \frac{2.3.3x^{\frac{9}{2}}}{2.4.2.4.9a^2} \text{ \&c.}$$

EXAM. 8.

To find the Fluent of $\dot{x}\sqrt{r^2x - sx^2}$.

1. $\sqrt{r^2x - sx^2}$ by *Ex.* 12, binomial theorem, is $= rx^{\frac{1}{2}} - \frac{sx^{\frac{3}{2}}}{2r} - \frac{s^2x^{\frac{5}{2}}}{4.2r^3} - \frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5} - \frac{3.5s^4x^{\frac{9}{2}}}{2.4.6.8r^7}$ &c.

2. Mult. by $\dot{x} = rx^{\frac{1}{2}}\dot{x} - \frac{sx^{\frac{3}{2}}}{2r}\dot{x} - \frac{s^2x^{\frac{5}{2}}}{2.4r^3}\dot{x} - \frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5}\dot{x} - \frac{3.5s^4x^{\frac{9}{2}}}{2.4.6.8r^7}\dot{x}$ &c.

3. Then the Fluent is as under by Rule 2.

1. The Fluent of $rx^{\frac{1}{2}}\dot{x} = \frac{rx^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}rx^{\frac{3}{2}}$.

2. —— of $-\frac{sx^{\frac{3}{2}}}{2r}\dot{x} = -\frac{sx^{\frac{5}{2}}}{\frac{5}{2}.2r} = -\frac{2sx^{\frac{5}{2}}}{5.2r}$
$= -\frac{sx^{\frac{5}{2}}}{5r}$.

3. —— of $-\frac{s^2x^{\frac{5}{2}}}{2.4r^3}\dot{x} = -\frac{s^2x^{\frac{7}{2}}}{\frac{7}{2}.2.4r^3} = -$
$\frac{2s^2x^{\frac{7}{2}}}{7.2.4r^3} = -\frac{s^2x^{\frac{7}{2}}}{7.4r^3}$.

4. —— of $-\frac{3s^3x^{\frac{7}{2}}}{2.4.6r^5}\dot{x} = -\frac{3s^3x^{\frac{9}{2}}}{\frac{9}{2}.2.4.6r^5} =$
$-\frac{3s^3x^{\frac{9}{2}}}{9.4.6r^5}$.

5. —— of $-\frac{3.5s^4x^{\frac{9}{2}}}{2.4.6.8r^7}\dot{x} = -\frac{3.5s^4x^{\frac{11}{2}}}{\frac{11}{2}.2.4.6.8r^7}$
$= -\frac{3.5s^4x^{\frac{11}{2}}}{11.4.6.8r^7}$.

Therefore Fluent $= \frac{2}{3}rx^{\frac{3}{2}} - \frac{sx^{\frac{5}{2}}}{5r} - \frac{s^2x^{\frac{7}{2}}}{7.4r^3} - \frac{3s^3x^{\frac{9}{2}}}{9.4.6r^5} - \frac{3.5s^4x^{\frac{11}{2}}}{11.4.6.8r^7}$ &c.

EXAM. 9.

To find the Fluent of this Fluxion $\frac{-ty\dot{y}}{c\sqrt{d^2-y^2}}$ *drawn into* $\frac{2s^3}{y} \times : \frac{y^2}{s^2} + \frac{1}{3} - \frac{s^2}{3.5y^2} + \frac{s^4}{5.7y^4} - \frac{s^6}{7.9y^6}$ &c.

Or,

To find the Fluent of $\frac{-ty\dot{y}}{c\sqrt{dd-yy}} \times \frac{2s^3}{y} \times : \frac{y^2}{ss} + \frac{1}{3} - \frac{s^2}{3.5y^2} + \frac{s^4}{5.7y^4} - \frac{s^6}{7.9y^6}$ &c.

Or,

Of $\frac{-2sty^2\dot{y}}{c\sqrt{dd-yy}} - \frac{2s^3t\dot{y}}{3c\sqrt{dd-yy}} + \frac{2s^5t\dot{y}}{3.5cy^2\sqrt{dd-yy}} - \frac{2s^7t\dot{y}}{5.7cy^4\sqrt{dd-yy}}$ &c.

Although each term may be found by Form 12, of Mr. *Emerſon*'s Table, yet, as there are a ſeries of terms, it will be ſhorter to find the Fluent by *Cor.* 3. *Prop.* IV. of Mr. *Emerſon*'s Fluxions, or *Prop.* XIV. following.

Let R $= \sqrt{dd-yy}$, then the Fl. $\frac{\dot{y}}{\sqrt{dd-yy}} = \varphi = \frac{\text{arch}}{d}$ of a circle, whoſe radius is d, and ſine y, and Fl. $\frac{y^2\dot{y}}{\sqrt{dd-yy}} = \frac{dd}{2}\varphi - \frac{y}{2}$ R. By *Cor.* 3, *Prop.* IV, we have this Theorem, Fl. $\frac{y^p\dot{y}}{R} =$

$y^{p+1}R$

$\frac{y^{p+1}R - \overline{p+2}.B}{\overline{p+1}.dd}$, where $m = -\frac{1}{2}$, $n = 2$, $e = dd$, $f = -1$, $z = y$, $B =$ Fl. of $\frac{y^{p+2}\dot{y}}{R}$.

Let α, β, γ, δ, &c. be the Fluents of $\frac{\dot{y}}{R}$, $\frac{\dot{y}}{y^2R}$, $\frac{\dot{y}}{y^4R}$, $\frac{\dot{y}}{y^6R}$ &c. then the ſucceſſive values

of p are -2, -4, -6, -8, &c.
and of B α, β, γ, δ, &c.

Hence from the above Theorem we have

$$\alpha = \varphi$$

$$\beta = \frac{y^{-1}R}{-dd} = \frac{-R}{ddy}$$

$$\gamma = \frac{y^{-3}R - 2\beta}{-3dd} = \frac{2\beta - \frac{R}{y^3}}{3dd}$$

$$\delta = \frac{y^{-5}R - 4\gamma}{-5dd} = \frac{4\gamma - \frac{R}{y^5}}{5dd}$$

$$\varepsilon = \frac{y^{-7}R - 6\delta}{-7dd} = \frac{6\delta - \frac{R}{y^7}}{7dd} \text{ \&c.}$$

Whence the Fluent $= \frac{-2st}{c} \times \frac{dd}{2}\varphi + \frac{styR}{2c} - \frac{s^3t}{3c}\alpha + \frac{s^5t}{3.5c}\beta - \frac{s^7t}{5.7c}\gamma$, &c.

EXAM. 10.

To find the Fluent of $\frac{2se^2 - 2sx^2}{a} \times : 1 - \frac{s^2a^2}{2.3 \times \overline{ee-xx}|^2} - \frac{s^4a^4}{2.4.5 \times \overline{ee-xx}|^4}$ *&c. drawn into* $\dot{x}$.

Or,

To find the Fluent of $\overline{ee-xx}\times\frac{2s\dot{x}}{a}-\frac{s^3a\dot{x}}{3.\overline{\overline{ee-xx}}}$ $-\frac{s^5a^3\dot{x}}{4.5.\overline{\overline{ee-xx}}|^3}$ &c.

By Form 1, the Fluent of $\overline{e^2-x^2}\times\frac{2s\dot{x}}{a}=\frac{2s}{a}\times$ $\overline{e^2x-\frac{1}{3}x^3}$; and by Form 6, the Fluent of $\frac{\dot{x}}{ee-xx}$ $=\frac{2.30258}{2e}\times\log.\frac{e+x}{e-x}$. Put α, β, γ, δ, &c. for the Fluents of $\frac{\dot{x}}{ee-xx}$, $\frac{\dot{x}}{\overline{\overline{ee-xx}}|^3}$, $\frac{\dot{x}}{\overline{\overline{ee-xx}}|^5}$, $\frac{\dot{x}}{\overline{\overline{ee-xx}}|^7}$, &c. and by Form 14, putting $\alpha=ee$, $\beta=-1$, $z=x$, $\eta=2$, $\pi=0$, $\delta=1+2\mu$, $\tau=2$, $\gamma=$ $ee-xx$, $\varphi=$ Fl. $\overline{e^2-x^2}|^{\mu}\dot{x}$, and we have Fl. $\overline{ee-xx}|^{\mu-2}\dot{x}=\frac{\delta.\overline{\delta-\eta}}{4e^4\mu.\overline{\mu-1}}\varphi-\frac{x\gamma^{\mu-1}}{\overline{\mu-1}.2ee}-$ $\frac{\overline{2\mu-1}.x\gamma^{\mu}}{\mu.\overline{\mu-1}.4e^4}$. Hence to find each Fluent β, γ, δ, &c. from the foregoing, the ſucceſſive values

of μ are -1, -3, -5, -7, &c.
of δ are -1, -5, -9, -13, &c.
and of φ are α, β, γ, δ, &c. whence by the foregoing Theorem in the laſt *Ex.* we have

$$\alpha=\frac{2.30258}{2e}\times\log.\frac{e+x}{e-x}$$

$$\beta=\frac{-1\times-3}{4e^4\times-1\times-2}\alpha-\frac{x\times\overline{ee-xx}|^{-2}}{-2\times2ee}-\frac{-3x\times\overline{ee-xx}|^{-1}}{4e^4\times-1\times-2}=\frac{1.3\alpha}{1.2.4e^4}+\frac{x}{4ee\times\overline{\overline{ee-xx}}|^2}+\frac{3x}{1.2.4e^4.\overline{\overline{ee-xx}}}$$

$\gamma=$

$$\gamma = \frac{-5\times-7\beta}{4e^4\times-3\times-4} - \frac{x\times\overline{ee-xx}|^{-4}}{-4.2ee} - \frac{-7x\times\overline{ee-xx}|^{-3}}{4e^4\times-3\times-4} = \frac{5.7\beta}{3.4.4e^4} + \frac{x}{8ee\times\overline{ee-xx}|^4} + \frac{7x}{3.4.4e^4\times\overline{ee-xx}|^3}$$

$$\delta = \frac{-9\times-11\gamma}{4e^4\times-5\times-6} - \frac{x\times\overline{ee-xx}|^{-6}}{-6\times2ee} - \frac{-11x\times\overline{ee-xx}|^{-5}}{4e^4\times-5\times-6} = \frac{9.11\gamma}{5.6.4e^4} + \frac{x}{12ee\times\overline{ee-xx}|^6} + \frac{11x}{5.6.4e^4\times\overline{ee-xx}|^5}\ \&c.$$

Whence the Fluent $= \frac{2se^2x}{a} - \frac{2sx^3}{3a} - \frac{s^3a}{3}\alpha - \frac{s^5a^3}{45}\beta$, &c.

EXAM. II.

Given the Fluxion $\overline{4z-4s}\times\frac{-tz\dot{z}}{c}\sqrt{\frac{zz-ss}{cc-zz}}$ $\times : \frac{1}{3} + \frac{z-s}{1.3.5.\overline{z+s}} - \frac{\overline{z-s}|^2}{3.5.7\overline{z+s}|^2}$ &c. $= \dot{P}$, *to find the Fluent.*

As this Fluent cannot be had but by infinite ſeries, firſt reduce $\sqrt{\frac{zz-ss}{cc-zz}}$ into a ſeries, which multiply by $\frac{-4tz^2\dot{z}+4stz\dot{z}}{c}$, then the reſulting ſeries multiplied ſeverally into the powers of $\frac{z-s}{z+s}$, reduced to a ſeries, after which the Fluent may be found. But we ſhall have the powers of z aſcending and deſcending both ways *ad infinitum*, therefore in my opinion it were better to transform it into another expreſſion, putting $z + s = v$, or $\frac{z-s}{z+s} = v$, &c. which will give a better ſeries; as the finding the Fluent of the above Fluxion is very difficult, I ſhall, for the ſake of my Reader, give the operation at large.

Put $v = \frac{z-s}{z+s}$, then $z = \frac{1+v}{1-v}\, s$; $\dot{z} = \frac{2s\dot{v}}{\overline{1-v}|^{2}}$; $4z - 4s = \frac{8sv}{1-v}$; $\frac{-tz\dot{z}}{c} = \frac{-2s^{2}t\dot{v}}{c} \times \frac{1+v}{\overline{1-v}|^{3}}$; $z^{2} - s^{2} = \frac{4s^{2}v}{\overline{1-v}|^{2}}$; $c^{2} - z^{2} = d^{2} \times \frac{1-2qv+v^{2}}{\overline{1-v}|^{2}}$, putting $d^{2} = c^{2} - s^{2}$, and $q = \frac{c^{2}+s^{2}}{c^{2}-s^{2}}$; hence $\sqrt{\frac{zz-ss}{cc-zz}} = 2s\sqrt{\frac{v}{1-2qv+v^{2}}}$, and the given Fluxion $\overline{4z-4s} \times \frac{-tz\dot{z}}{c}\sqrt{\frac{zz-ss}{cc-zz}} \times : \frac{1}{3} + \frac{1}{3\,5} \times \frac{z-s}{z+s} - \frac{1}{3\cdot5\cdot7} \times \overline{\frac{z-s}{z+s}}|^{2}$ &c. $= \frac{-32s^{4}t\dot{v}v}{cd} \times \frac{1+v}{\overline{1-v}|^{4}}\sqrt{\frac{v}{1-2qv+v^{2}}} \times : \frac{1}{3} + \frac{v}{3\cdot5} - \frac{vv}{3\cdot5\cdot7}$ &c.

Now $\sqrt{\frac{v}{1-2qv+vv}} = v^{\frac{1}{2}} \times : 1 + qv + \frac{3q^{2}-1}{2}v^{2} + \frac{5q^{3}-3q}{2}v^{3}$ &c. which divided by $\overline{1-v}|^{4}$ and multiplied by $1+v$, thus

$$1 - 4v + 6v^{2} - 4v^{3} + v^{4})\ 1 + qv + \frac{3qq-1}{2}v^{2} + \frac{5q^{3}-3q}{2}v^{3}\ (1 + q + 4.v + \frac{3qq+8q+19}{2}v^{2} + \frac{5q^{3}+12q^{2}+17q^{2}+36}{2}v^{3}$$ &c. quotient.

And $1 + \overline{q + 4}.v + \frac{3q^2 + 8q + 19}{2} v^2 + \frac{5q^3 + 12q^2 + 17q + 36}{2} v^3$ &c.

Mult. by $1 + v$

Product $1 + \overline{q + 5}.v + \frac{3q^2 + 10q + 27}{2} v^2 + \frac{5q^3 + 15q^2 + 25q + 55}{2} v^3$ &c.

And mult. by $\frac{1}{3} + \frac{v}{3.5} - \frac{v^2}{3.5.7}$ &c.

Product $= \frac{1}{3} \times : 1 + \frac{5q + 26}{5} v + \frac{105q^2 + 364q + 1017}{2.5.7} v^2$ &c.

And ſo the original Fluxion becomes $\frac{-32s^4tv^{\frac{3}{2}}\dot{v}}{3cd} \times : 1 + \frac{26 + 5q}{5} v + \frac{1017 + 364q + 105q^2}{2.5.7} v^2$ &c. whoſe Fluent is $\frac{-32s^4tv^{\frac{5}{2}}}{cd} \times : \frac{2}{3.5} + \frac{52 + 10q}{3.5.7} v + \frac{1017 + 364q + 105q^2}{3.5.7.9} v^2$ &c. $= \mathrm{P}$, as required.

EXAM. 12.

Given the Fluxion $\overline{de^2 - bx^2}|^{\frac{3}{2}} \times \overline{se^2 - bx^2}|^{\frac{1}{2}} \times \frac{4\dot{x}}{3e^4} + \frac{4}{3.5} \times \frac{\overline{de^2 - bx^2}|^{\frac{5}{2}}}{\overline{se^2 - bx^2}|^{\frac{1}{2}}} \times \frac{\dot{x}}{e^4}$ &c. $= \dot{\mathrm{Q}}$, *to find the Fluent ?*

Put $\frac{de^2 - bx^2}{se^2 - bx^2} = v$, hence $bx^2 = \frac{d - sv}{1 - v} e^2$, conſequently $se^2 - bx^2 = \frac{s - d}{1 - v} e^2$; $de^2 - bx^2 = \frac{s - d}{1 - v} ve^2$.

Alſo

expreſſion, putting $z + s = v$, or $\frac{z-s}{z+s} = v$, &c. which will give a better ſeries; as the finding the Fluent of the above Fluxion is very difficult, I ſhall, for the ſake of my Reader, give the operation at large.

Put $v = \frac{z-s}{z+s}$, then $z = \frac{1+v}{1-v}\, s$; $\dot{z} = \frac{2s\dot{v}}{\overline{1-v}|^{2}}$; $4z - 4s = \frac{8sv}{1-v}$; $\frac{-tz\dot{z}}{c} = \frac{-2s^2t\dot{v}}{c} \times \frac{1+v}{\overline{1-v}|^{3}}$; $z^2 - s^2 = \frac{4s^2v}{\overline{1-v}|^{2}}$; $c^2 - z^2 = d^2 \times \frac{1-2qv+v^2}{\overline{1-v}|^{2}}$, putting $d^2 = c^2 - s^2$, and $q = \frac{c^2+s^2}{c^2-s^2}$; hence $\sqrt{\frac{zz-ss}{cc-zz}} = 2s\sqrt{\frac{v}{1-2qv+v^2}}$, and the given Fluxion $\overline{4z-4s} \times \frac{-tz\dot{z}}{c}\sqrt{\frac{zz-ss}{cc-zz}} \times : \frac{1}{3} + \frac{1}{3\ 5} \times \frac{z-s}{z+s} - \frac{1}{3.5.7} \times \overline{\frac{z-s}{z+s}}\Big|^{2}$ &c. $= \frac{-32s^4tv\dot{v}}{cd} \times \frac{1+v}{\overline{1-v}|^{4}}\sqrt{\frac{v}{1-2qv+v^2}} \times : \frac{1}{3} + \frac{v}{3.5} - \frac{vv}{3.5.7}$ &c.

Now $\sqrt{\frac{v}{1-2qv+vv}} = v^{\frac{1}{2}} \times : 1 + qv + \frac{3q^2-1}{2}v^2 + \frac{5q^3-3q}{2}v^3$ &c. which divided by $\overline{1-v}|^{4}$ and multiplied by $1+v$, thus

$$1 - 4v + 6v^2 - 4v^3 + v^4)\ 1 + qv + \frac{3qq-1}{2}v^2 + \frac{5q^3-3q}{2}v^3\ (\overline{1 + q+4}.v + \frac{3qq+8q+19}{2}v^2 + \frac{5q^3+12q^2+17q^2+36}{2}v^3 \text{ \&c. quotient.}$$

And $1+\overline{q+4}\cdot v+\frac{3q^2+8q+19}{2}v^2+\frac{5q^3+12q^2+17q+36}{2}v^3$ &c.

Mult. by $1+v$

Product $1+\overline{q+5}\cdot v+\frac{3q^2+10q+27}{2}v^2+\frac{5q^3+15q^2+25q+55}{2}v^3$ &c.

And mult. by $\frac{1}{3}+\frac{v}{3\cdot5}-\frac{v^2}{3\cdot5\cdot7}$ &c.

Product $=\frac{1}{3}\times:1+\frac{5q+26}{5}v+\frac{105q^2+364q+1017}{2.5\cdot7}v^2$ &c.

And ſo the original Fluxion becomes

$\frac{-32s^4tv^{\frac{3}{2}}\dot{v}}{3cd}\times:1+\frac{26+5q}{5}v+\frac{1017+364q+105q^2}{2\cdot5\cdot7}v^2$ &c. whoſe Fluent is $\frac{-32s^4tv^{\frac{5}{2}}}{cd}\times:\frac{2}{3\cdot5}+\frac{52+10q}{3\cdot5\cdot7}v+\frac{1017+364q+105q^2}{3\cdot5\cdot7\cdot9}v^2$ &c. $=\text{P}$, as required.

EXAM. 12.

Given the Fluxion $\overline{de^2-bx^2}|^{\frac{3}{2}}\times\overline{se^2-bx^2}|^{\frac{1}{2}}\times\frac{4\dot{x}}{3e^4}+\frac{4}{3\cdot5}\times\frac{\overline{de^2-bx^2}|^{\frac{5}{2}}}{\overline{se^2-bx^2}|^{\frac{1}{2}}}\times\frac{\dot{x}}{e^4}$ &c. $=\dot{\text{Q}}$, *to find the Fluent?*

Put $\frac{de^2-bx^2}{se^2-bx^2}=v$, hence $bx^2=\frac{d-sv}{1-v}e^2$, conſequently $se^2-bx^2=\frac{s-d}{1-v}e^2$; $de^2-bx^2=\frac{s-d}{1-v}ve^2$.

Alſo $\dot{x}$ = Fluxion of $\sqrt{\frac{d-sv}{1-v} \times \frac{e^2}{b}} = \frac{e\dot{v}}{2\sqrt{b}} \times \frac{d-s}{\overline{1-v}\sqrt{\overline{1-v} \times \overline{d-sv}}}$; and the Fluxion becomes $\frac{-2ep^3v^{\frac{3}{2}}\dot{v}}{3.\overline{1-v}|^3.\sqrt{\overline{1-v}.\overline{d-sv}}} \times \overline{\frac{1}{n} + \frac{v}{5}}$ &c. (putting $p = s - d$, and $n = \sqrt{b}$).

Now $\overline{1-v}.\overline{d-sv} = d - sv - dv + sv^2 = d \times \overline{1 - \overline{1 + \frac{s}{d}} \times v + \frac{s}{d} v^2}$.

And $\sqrt{1 - \overline{1 + \frac{s}{d}} \times v + \frac{s}{d} v^2} = 1 - \frac{s+d}{2d} v - \frac{\overline{s-d}|^2}{8d^2} v^2$ &c.

N. B. $\frac{\overline{s-d}|^2}{4d^2}$ is $= \frac{\overline{s+d}|^2}{4dd} - \frac{s}{d}$.

And $1 - \frac{s+d}{2d} v - \frac{\overline{s-d}|^2}{8d^2} v^2)\ 1\ (1 + \frac{s+d}{2d} v + \frac{\overline{s+d}|^2 + 2.s^2 + d^2}{8d^2} v^2$ &c. which divided by $\overline{1-v}|^3$, thus

$1 - 3v + 3v^2 - v^3)\ 1 + \frac{s+d}{2d} v + \frac{\overline{s+d}|^2 + 2.\overline{s^2+d^2}}{8d^2} v^2\ (1 + \frac{s+7d}{2d} v + \frac{\overline{s+7d}|^2 + 2\ \overline{s^2+7d^2}}{8d^2} v^2$, &c. quotient.

And $1 + \frac{s+7d}{2d} v + \frac{\overline{s+7d}|^2 + 2.\overline{s^2+7d^2}}{8d^2} v^2$ &c.

Mult. by $\frac{1}{n} + \frac{1}{5} v$

Prod. $= \frac{1}{n} + \overline{\frac{1}{5} + \frac{s+7d}{2nd}} \times v$; then the Fluxion becomes $\frac{-2e.\overline{s-d^3v}|^{\frac{3}{2}}\dot{v}}{3\sqrt{d}} \times : \overline{\frac{1}{\sqrt{b}} + \frac{1}{5} + \frac{s+7d}{2d\sqrt{b}}.v}$ &c. and the Fluent is $\frac{-2e.\overline{s-d}|^3v^{\frac{5}{2}}}{3\sqrt{d}} \times \frac{2}{5\cdot7} + \frac{s+7d}{7d\sqrt{b}}v$ &c. $=$ Q, which was required.

I here acknowledge my obligation to Mr. profeſſor *Hutton*, of *Woolwich*, for aſſiſting me in the above Fluent, for which he has my thanks.

RULE. 8.

Sometimes the Fluent may be found by aſſuming other variable quantities to make up the Fluent, and finding their values by help of the given equations and their Fluxions. See Mr. *Emerſon*'s Fluxions, p. 63.

EXAM. 13.

To find the Fluent of $\dot{x}\sqrt{dx-xx}$?

Let $d-x=v$, and $-\dot{x}=\dot{v}$; then $\dot{x}\sqrt{dx-xx} = \dot{x}\sqrt{vx} = v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$. Suppoſe Fl. $v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} = \frac{2}{3}v^{\frac{1}{2}}x^{\frac{3}{2}} + s$, then will $v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} = v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} + \frac{x^{\frac{3}{2}}\dot{v}}{3v^{\frac{1}{2}}} + \dot{s}$, or $\dot{s} = \frac{-x^{\frac{3}{2}}\dot{v}}{3v^{\frac{1}{2}}} = \frac{+x^{\frac{3}{2}}\dot{x}}{3v^{\frac{1}{2}}}$. Suppoſe again $s = \frac{2x^{\frac{5}{2}}}{3\cdot5v^{\frac{1}{2}}} + t$, then will $\dot{t} = \frac{x^{\frac{5}{2}}\dot{v}}{3\cdot5v^{\frac{3}{2}}} = \frac{-x^{\frac{5}{2}}\dot{x}}{3\cdot5v^{\frac{3}{2}}}$, and $t = \frac{-2x^{\frac{7}{2}}}{3\cdot5\cdot7v^{\frac{3}{2}}} + u$, then will $\dot{u} = \frac{x^{\frac{7}{2}}\dot{v}}{3\cdot5\cdot7v^{\frac{3}{2}}}$, and $u = \frac{2x^{\frac{9}{2}}}{5\cdot7\cdot9v^{\frac{5}{2}}} +$ &c. whence the Fluent of $\dot{x}\sqrt{dx-xx} = \frac{2}{3}v^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{2x^{\frac{5}{2}}}{3\cdot5v^{\frac{1}{2}}} - \frac{2x^{\frac{7}{2}}}{3\cdot5\cdot7v^{\frac{3}{2}}} + \frac{2x^{\frac{9}{2}}}{5\cdot7\cdot9v^{\frac{5}{2}}}$ &c. $= 2\sqrt{vx} \times :$ $\frac{x}{1\cdot3} + \frac{x^2}{3\cdot5v} - \frac{x^3}{3\cdot5\cdot7v^2} + \frac{x^4}{5\cdot7\cdot9v^3} - \frac{x^5}{7\cdot9\cdot11v^4}$ &c.

EXPLANATION.

1. The Fluent of $v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} = \frac{v^{\frac{1}{2}}x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2v^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$ by the Rule.

2. Fluent of $\frac{x^{\frac{3}{2}}\dot{x}}{3v^{\frac{1}{2}}} = \frac{x^{\frac{5}{2}}}{\frac{5}{2}\cdot 3v^{\frac{1}{2}}} = \frac{2x^{\frac{5}{2}}}{3\cdot 5v^{\frac{1}{2}}}$.

3. Fluent of $\frac{-x^{\frac{5}{2}}\dot{x}}{3\cdot 5v^{\frac{3}{2}}} = \frac{-x^{\frac{7}{2}}}{\frac{7}{2}\cdot 3\cdot 5v^{\frac{3}{2}}} = -\frac{2x^{\frac{7}{2}}}{3\cdot 5\cdot 7v^{\frac{3}{2}}}$ &c.

EXAM. 14.

To find the Fluent of $\dot{x}\sqrt{tx+xx}$, *or the quadrature of the right angled hyperbola.*

Let $t + x = v$, then in Fluxions $\dot{x} = \dot{v}$, and $\dot{x}\sqrt{tx+xx} = v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$.

Let Fluent $v^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} = \frac{2}{3}v^{\frac{1}{2}}x^{\frac{3}{2}} + s$, in Fluxions $\dot{s} = \frac{-x^{\frac{3}{2}}v}{3v^{\frac{1}{2}}} = \frac{-x^{\frac{3}{2}}\dot{x}}{3v^{\frac{1}{2}}}$ and $s = \frac{-2x^{\frac{5}{2}}}{3\cdot 5v^{\frac{1}{2}}} + t$, in Fluxions $\dot{t} = \frac{-x^{\frac{5}{2}}\dot{x}}{3\cdot 5v^{\frac{3}{2}}}$, and $t = \frac{-2x^{\frac{7}{2}}}{3\cdot 5\cdot 7v^{\frac{3}{2}}} + u$, and $\dot{u} = \frac{-x^{\frac{7}{2}}\dot{x}}{5\cdot 7v^{\frac{5}{2}}}$, and $u = \frac{-2x^{\frac{9}{2}}}{5\cdot 7\cdot 9v^{\frac{5}{2}}}$ &c. whence Fluent

$$\dot{x}\sqrt{tx+xx} = \frac{2}{3}v^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{3\cdot 5v^{\frac{1}{2}}} - \frac{2x^{\frac{7}{2}}}{3\cdot 5\cdot 7v^{\frac{3}{2}}} - \frac{2x^{\frac{9}{2}}}{5\cdot 7\cdot 9v^{\frac{5}{2}}} \text{ \&c.} = 2\sqrt{vx} \times : \frac{x}{1\cdot 3} - \frac{x^2}{1\cdot 3\cdot 5v} - \frac{x^3}{3\cdot 5\cdot 7v^2} - \frac{x^4}{5\cdot 7\cdot 9v^3} \text{ \&c.}$$

Theſe two laſt ſeries are thoſe given in *Shirtliff*'s Gauging, without their inveſtigation.

EXAM. 15.

To find the Fluent of $\dot{x}x^{n-1}$.

1. $\dot{x}x^n \times z^{-1} = r$. Aſſume $\frac{x^{n+1}}{n+1} \times z^{-1} + s = r$, and find each Fluent. Then $\dot{r} = \dot{x}x^n z^{-1} - \dot{z}z^{-2} \times \frac{x^{n+1}}{n+1} + \dot{s} = \dot{x}x^n \times z^{-1}$.

Therefore $\dot{s} - \dot{z}z^{-2} \times \frac{x^{n+1}}{n+1} = 0$. Therefore $\dot{s} = \frac{x^{n+1}}{n+1} \times \dot{z}z^{-2}$.

Therefore $\dot{s} = \frac{x^{n+1}}{n+1} \dot{x}z^{-2}$, by writing $\dot{x}$ for $\dot{z}$; aſſume $s = \frac{-x^{n+2}}{n+1.n+2} z^{-2} + w$.

Then $\dot{s} = \frac{\dot{x}x^{n+2}}{n+1} z^{-2} - \frac{2x^{n+2}}{n+1.n+2} \dot{z}z^{-3} + \dot{w} = \frac{\dot{x}x^{n+1}}{n+1} z^{-2}$.

Therefore $\dot{w} - \frac{2x^{n+2}}{n+1.n+2} \dot{z}z^{-3} = 0$; and $\dot{w} = \frac{2\dot{x}x^{n+2}}{n+1.n+2} z^{-3}$.

Aſſume $\frac{2x^{n+3}}{n+1.n+2.n+3} z^{-3} + v = w$.

Fluent of $\dot{x}x^n \times z^{-1} = \dot{x}x^{n-1}$ is $\frac{x^{n+1}z^{-1}}{n+1} + \frac{x^{n+2}z^{-2}}{n+1.n+2} + \frac{2x^{n+3}}{n+1.n+2.n+3} z^{-3}$ &c. $= \frac{x^{n+1}z^{-1}}{n} = \frac{x^n}{n}$, and when $x = 1$, the Fluent is $\frac{1}{n+1} +$

$$\frac{1}{n+1.n+2}+\frac{1.2}{n+1.n+2.n+3}+\frac{1.2.3}{n+1.n+2.n+3.n+4}\ \&c.=\frac{1}{n}.$$

Or thus.

Assume the Fluent of $\dot{x}x^{n-1}$ or $\dot{x}x^{n}x^{-1}=\frac{x^{n+1}}{n+1}x^{-1}+s$, this in Fluxions gives $\dot{x}x^{n-1}=\dot{x}x^{n-1}-\frac{x^{n+1}}{n+1}x^{-2}\dot{x}+\dot{s}$, whence $\dot{s}=\frac{x^{n+1}}{n+1}x^{-2}\dot{x}$.

Again, assume $s=\frac{x^{n+2}}{n+1.n+2}x^{-2}+t$, whence $\dot{t}=\frac{2x^{n+2}}{n+1.n+2}x^{-3}\dot{x}$; assume $t=\frac{2x^{n+3}}{n+1.n+2.n+3}x^{-3}+u$, &c. whence we have the Fluent $\dot{x}x^{n-1}=\frac{x^{n+1}}{\overline{n+1}.x}+\frac{x^{n+2}}{\overline{n+1}.\overline{n+2}.x^{2}}+\frac{2.x^{n+3}}{\overline{n+1}\ldots\overline{n+3}.x^{3}}+\frac{2.3.x^{n+4}}{\overline{n+1}\ldots\overline{n+4}.x^{4}}+\&c.=\frac{x^{n}}{n}$ as before.

EXAM. 16.

To find the Fluent of $\frac{n\dot{x}x}{r}^{\frac{m-n-2r}{r}}$?

Assume Fluent $\frac{n\dot{x}}{r}x^{\frac{m-n-2r}{r}}=\frac{n}{m}x^{\frac{m-n-r}{r}}+s=\frac{n}{m}x^{\frac{m}{r}-\frac{n+r}{r}}+s$, in Fluxions $\frac{n\dot{x}}{r}x^{\frac{m-n-2r}{r}}=\frac{n\dot{x}}{r}x^{\frac{m-n-2r}{r}}-\frac{n+r}{r}\times\frac{n}{m}x^{\frac{m-n-2r}{r}}\dot{x}+\dot{s}$, whence $\dot{s} = \frac{n.\overline{m+r}}{mr}\dot{x}x^{\frac{m-n-2r}{r}} = \frac{n.\overline{n+r}}{rm}x^{\frac{m}{r}}$ $x^{-\frac{n+2r}{r}}\dot{x}$; aſſume $s = \frac{n.\overline{n+r}}{m.\overline{m+r}}x^{\frac{m+r}{r}-\frac{n+2r}{r}} + t$, whence $\dot{t} = \frac{n.\overline{n+r}.\overline{n+2r}}{r.m.\overline{m+r}}x^{\frac{m-n-2r}{r}}\dot{x} =$ $\frac{n.\overline{n+r}.\overline{n+2r}}{rm.\overline{m+r}}x^{\frac{m+r}{r}-\frac{n+3r}{r}}\dot{x}$.

Aſſume $t = \frac{n.\overline{n+r}.\overline{n+2r}}{m.\overline{m+r}.\overline{m+2r}}x^{\frac{m+2r}{r}-\frac{n+3r}{r}} + u$,

Whence $\dot{u} = \frac{n\ldots\overline{n+3r}}{rm\ldots\overline{m+2r}}\dot{x}x^{\frac{m+2r}{r}}x^{-\frac{n+4r}{r}}$.

Aſſume $u = \frac{n\ldots\ldots\overline{n+3r}}{m\ldots\ldots\overline{m+3r}}x^{\frac{m+3r}{r}}x^{-\frac{n+4r}{r}} + w$ &c.

Hence Fluent $\frac{n\dot{x}}{r}x^{\frac{m-n-2r}{r}} = \frac{n}{m}x^{\frac{m-n-r}{r}} +$ $\frac{n.\overline{n+r}}{m.\overline{m+r}}x^{\frac{m-n-r}{r}} + \frac{n\ldots\overline{n+2r}}{m\ldots\overline{m+2r}}x^{\frac{m-n-r}{r}} +$ $\frac{n\ldots\ldots\overline{n+3r}}{m\ldots\overline{m+3r}}x^{\frac{m-n-r}{r}}$ &c. $= \frac{n}{m-n-r}x^{\frac{m-n-r}{r}}$, and when $x = 1$, the Fluent is $= \frac{n}{m} + \frac{n.\overline{n+r}}{m.\overline{m+r}} +$ $\frac{n.\overline{n+r}.\overline{n+2r}}{m.\overline{m+r}.\overline{m+2r}} + \frac{n.\overline{n+r}.\overline{n+2r}.\overline{n+3r}}{m.\overline{m+r}\,\overline{m+2r}.\overline{m+3r}}$ &c. $=$ $\frac{n}{m-n-r}$.

Or thus.

The ſeries may alſo be deduced from another fluxionary quantity, viz. $\dot{x}x^{m-1-n-r}$, putting $x = 1$, thus

Aſſume Fl. $\dot{x}x^{m-1-n-r} = \frac{x^m}{m}x^{-n-r}. + s$, then will it be in Fluxions $\dot{s} = \frac{\overline{n+r}.x^m x^{-n-r-1}\dot{x}}{m}$, aſſume $s = \frac{\overline{n+r}.x^{m+r}x^{-n-2r}}{m.\overline{m+r}} + t$, then $\dot{t} = \frac{\overline{n+r}.\overline{n+2r}}{m.\overline{m+r}}\dot{x}x^{m+2r-1}x^{-n-3r}$;

Aſſume $t = \frac{\overline{n+r}.\overline{n+2r}}{m.\overline{m+r}.\overline{m+2r}}x^{m+2r}x^{-n-3r} + u$ &c. whence the Fluent $\dot{x}x^{m-1-n-r} = \frac{x^m}{mx^{n+r}} + \frac{\overline{n+r}.x^{m+r}}{m.\overline{m+r}.x^{n+2r}} + \frac{\overline{n+r}.\overline{n+2r}.x^{m+2r}}{m.\overline{m+r}.\overline{m+2r}.x^{n+3r}}$ &c. $= \frac{x^{m-n-r}}{m-n-r}$, when $x = 1$, multiply all by n, and we have the ſeries required. And, by a like method, infinite forms may be found.

EXAM. 17.

To find the Fluent of $\frac{m-r.a\dot{x}x^{m-n-2r}}{r}$?

Aſſume Fl. $\frac{m-r}{r}a\dot{x}x^{\frac{m-n-2r}{r}} = ax^{\frac{m-n-r}{r}} + s$ $= ax^{\frac{m-r}{r}}x^{-\frac{n}{r}} + s$,

Whence $\dot{s} = \frac{na}{r}\dot{x}x^{\frac{m-r}{r} - \frac{n+r}{r}}$,

Aſſume $s = \frac{an}{m}x^{\frac{m}{r} - \frac{n+r}{r}} + t$; whence $\dot{t} = \frac{an.\overline{n+r}}{rm}x^{\frac{m}{r} - \frac{n+2r}{r}}\dot{x}$, Aſſume $t = \frac{an.\overline{n+r}}{m.\overline{m+r}} x^{\frac{m+r}{r} - \frac{n+2r}{r}} + u$; then

$\dot{u} = \frac{an.\overline{n+2r}}{rm.\overline{m+r}} \dot{x}x^{\frac{m+r}{r} - \frac{n+3r}{r}}$;

Aſſume $u = \frac{an\ldots\overline{n+2r}}{m\ldots\overline{m+2r}} x^{\frac{m+2r}{r} - \frac{n+3r}{r}} + w$,

Hence $\dot{w} = \frac{an\ldots\overline{n+3r}}{rm\ldots\overline{m+2r}} \dot{x}x^{\frac{m+2r}{r} - \frac{n+4r}{r}}$,

Aſſume $w = \frac{an\ldots\overline{n+3r}}{m\ldots\overline{m+3r}} x^{\frac{m+3r}{r} - \frac{m+4r}{r}}$ &c.

Hence Fl. $\frac{m-r}{r} a\dot{x}x^{\frac{\overline{m-n-2r}}{r}} = ax^{\frac{\overline{m-n-r}}{r}} + \frac{an}{m} x^{\frac{m-n-r}{r}} + \frac{an.\overline{n+r}}{m.\overline{m+r}} x^{\frac{m-n-r}{r}} + \frac{an\ldots\overline{n+2r}}{m\ldots\overline{m+2r}} x^{\frac{m-n-r}{r}} + \frac{an\ldots\overline{n+3r}}{m\ldots\overline{m+3r}} x^{\frac{m-n-r}{r}} + \&c. = \frac{m-r}{m-n-r} ax^{\frac{\overline{m-n-r}}{r}}$. And when $x = 1$, the Fluent of $\frac{\overline{m-r}.a\dot{x}x^{\frac{\overline{m-n-2r}}{r}}}{r}$ is $= a + \frac{an}{m} + \frac{a.n.\overline{n+r}}{m.\overline{m+r}} + \frac{a.n.\overline{n+r}.\overline{n+2r}}{m.\overline{m+r}.\overline{m+2r}} + \frac{a.n.\overline{n+r}.\overline{n+2r}.\overline{n+3r}}{m.\overline{m+r}.\overline{m+2r}.\overline{m+3r}}$ &c. $= \frac{a.\overline{m-r}}{m-n-r}$.

I ſhall now illuſtrate five general Propoſitions of the univerſally learned Mr. EMERSON in his Doctrine of Fluxions, which will be found to be of admirable uſe in finding and comparing of Fluents.

PROP. XIV.

Let the Fluent of $\overline{e+fz^n}|^{m}z^p\dot{z} = \text{A}$.
Fluent of $\overline{e+fz^n}|^{m}z^{p+n}\dot{z} = \text{B}$.
Fluent of $\overline{e+fz^n}|^{m+1}z^p\dot{z} = \text{P}$.

Then I ſay,

I. $\overline{p+1}.\text{P}+\overline{m+1}\times nf\text{B} = z^{p+1}\overline{e+fz^n}|^{m+1}$.
II. $\text{P} = e\text{A} + f\text{B}$.

This is Mr. *Emerſon*'s 4th Prop. p. 15.

ILLUSTRATION.

EXAM. 1.

Let $n = 1$, $m = \frac{1}{2}$, $p = 1$, *whence*
Fl. $\overline{e+fz}|^{\frac{1}{2}}z\dot{z} = \text{A}$.
Fl. $\overline{e+fz}|^{\frac{1}{2}}z^2\dot{z} = \text{B}$.
Fl. $\overline{e+fz}|^{\frac{3}{2}}z\dot{z} = \text{P}$.

Then,

I. $2\text{P} + \frac{3}{2}f\text{B}$, or 2 Fl. $\overline{e+fz}|^{\frac{3}{2}}z\dot{z} + \frac{3}{2}f \times$ Fl. $\overline{e+fz}|^{\frac{1}{2}}z^2\dot{z} = z^2 \times \overline{e+fz}|^{\frac{3}{2}}$.

II. $\text{P} = e\text{A} + f\text{B}$, or Fl. $\overline{e+fz}|^{\frac{3}{2}}z\dot{z} = e \times$ Fl. $\overline{e+fz}|^{\frac{1}{2}} z\dot{z} + f \times$ Fl. $\overline{e+fz}|^{\frac{1}{2}}z^2\dot{z}$.

COR. 1. Hence if any one of the Fluents, A, B, P, be given, the other two will be found. Therefore,

COR. 2. $\text{B} = \dfrac{z^{p+1}.\overline{e+fz^n}|^{m+1} - \overline{p+1}\times e\text{A}}{\overline{p+1+mn+n}\times f}$.

COR. 3. $\text{A} = \dfrac{z^{p+1}.\overline{e+fz^n}|^{m+1} - \overline{p+1+mn+n}.f\text{B}}{\overline{p+1}\times e}$.

COR. 4. $\text{P} = \dfrac{\overline{m+1}.ne\text{A} + z^{p+1}\overline{e+fz^n}|^{m+1}}{p+1+mn+n}$.

COR. 5. $A = \frac{\overline{p+1+mn+n}.P - z^{p+1}\overline{e+fz^n}|^{m+1}}{\overline{m+1}\times ne}$.

EXAM. 2.

Let $n = 2$, $m = \frac{1}{2}$, $p = 1$, *whence*

$$Fl.\ \overline{e+fzz}|^{\frac{1}{2}}z\dot{z} = A.$$

$$Fl.\ \overline{e+fzz}|^{\frac{1}{2}}z^3\dot{z} = B.$$

$$Fl.\ \overline{e+fzz}|^{\frac{3}{2}}z\dot{z} = P.$$

Then,

I. $2P + \frac{3}{2} \times 2fB$, or $2\ \text{Fl.}\ z\dot{z}\,\overline{e+fzz}|^{\frac{3}{2}} + \frac{3}{2} \times 2f \times \text{Fl.}\ \overline{e+fzz}|^{\frac{1}{2}}z^3\dot{z} = z^2.\overline{e+fzz}|^{\frac{3}{2}}$.

II. $P = eA + fB$, or $\overline{e+fzz}|^{\frac{3}{2}}z\dot{z} = e \times \text{Fl.}\overline{e+fzz}|^{\frac{1}{2}}z\dot{z} + f \times \text{Fl.}\ \overline{e+fzz}|^{\frac{1}{2}}z^3\dot{z}$.

EXAM. 3.

Let $n = 3$, $m = \frac{1}{2}$, $p = 1$, *whence*

$$Fl.\ \overline{e+fz^3}|^{\frac{1}{2}}z\dot{z} = A.$$

$$Fl.\ \overline{e+fz^3}|^{\frac{1}{2}}z^4\dot{z} = B.$$

$$Fl.\ \overline{e+fz^3}|^{\frac{3}{2}}z\dot{z} = P.$$

Then,

I. $2P + \frac{3}{2} \times 3fB$, or $2\ \text{Fl.}\ \overline{e+fz^3}|^{\frac{3}{2}}z\dot{z} + \frac{3}{2} \times 3f \times \text{Fl.}\ \overline{e+fz^3}|^{\frac{1}{2}}z^4\dot{z} = zz.\overline{e+fz^3}|^{\frac{3}{2}}$.

II. $P = eA + fB$, or $\text{Fl.}\ \overline{e+fz^3}|^{\frac{3}{2}}z\dot{z} = e \times \text{Fl.}\ \overline{e+fz^3}|^{\frac{1}{2}}z\dot{z} + f \times \text{Fl.}\ \overline{e+fz^3}|^{\frac{1}{2}}z^4\dot{z}$.

EXAM. 4.

Let $n = 4$, $m = \frac{1}{2}$, $p = 1$, *whence*

$$Fl.\ \overline{e+fz^4}|^{\frac{1}{2}}z\dot{z} = A.$$

$$Fl.\ \overline{e+fz^4}|^{\frac{1}{2}}z^5\dot{z} = B.$$

$$Fl.\ \overline{e+fz^4}|^{\frac{3}{2}}z\dot{z} = P.$$

Then,

I. $2P+\frac{3}{2}\times 4fB$, or 2 Fl. $z\dot{z}.\overline{e+fz^4}|^{\frac{3}{2}}+\frac{3}{2}\times 4f\times$ Fl. $\overline{e+fz^4}|^{\frac{1}{2}}z^5\dot{z}=z^2.\overline{e+fz^4}|^{\frac{3}{2}}$.

II. $P=eA+fB$, or Fl. $\overline{e+fz^4}|^{\frac{3}{2}}z\dot{z}=e\times\overline{e+fz^4}|^{\frac{1}{2}}z\dot{z}+f\times$ Fl. $\overline{e+fz^4}|^{\frac{1}{2}}z^5\dot{z}$.

EXAM. 5.

Let $n=1$, $m=\frac{3}{2}$, $p=2$, *whence*

Fl. $\overline{e+fz}|^{\frac{3}{2}}z^2\dot{z}=A$.

Fl. $\overline{e+fz}|^{\frac{3}{2}}z^3\dot{z}=B$.

Fl. $\overline{e+fz}|^{\frac{5}{2}}z^2\dot{z}=P$.

Then,

I. $3P+\frac{5}{2}\times fB=z^{p+1}\overline{e+fz^n}|^{m+1}$, or 3 Fl. $\overline{e+fz}|^{\frac{5}{2}}z^2\dot{z}+\frac{5}{2}\times f\times$ Fl. $\overline{e+fz}|^{\frac{3}{2}}z^3\dot{z}=z^3.\overline{e+fz}|^{\frac{5}{2}}$.

II. $P=eA+fB$, or $\overline{e+fz}|^{\frac{5}{2}}z^2\dot{z}=e\times$ Fl. $\overline{e+fz}|^{\frac{3}{2}}z^2\dot{z}+f\times$ Fl. $\overline{e+fz}|^{\frac{3}{2}}z^3\dot{z}$.

EXAM. 6.

Let $n=3$, $m=\frac{5}{2}$, $p=2$, *whence*

Fl. $\overline{e+fz^3}|^{\frac{5}{2}}z^2\dot{z}=A$.

Fl. $\overline{e+fz^3}|^{\frac{5}{2}}z^5\dot{z}=B$.

Fl. $\overline{e+fz^3}|^{\frac{7}{2}}z^2\dot{z}=P$.

Then,

I. $3P+\frac{7}{2}\times 3fB=z^{p+1}\overline{e+fz^n}|^{m+1}$, or 3 Fl. $\overline{e+fz^3}|^{\frac{7}{2}}z^2\dot{z}+\frac{7}{2}\times 3f\times$ Fl. $\overline{e+fz^3}|^{\frac{5}{2}}z^5\dot{z}=z^3.\overline{e+fz^3}|^{\frac{7}{2}}$.

II. $P=eA+fB$, or Fl. $\overline{e+fz^3}|^{\frac{7}{2}}z^2\dot{z}=e\times$ Fl. $\overline{e+fz^3}|^{\frac{5}{2}}z^2\dot{z}+f\times$ Fl. $\overline{e+fz^3}|^{\frac{5}{2}}z^5\dot{z}$.

EXAM. 7. More general.

Let $n=3\frac{1}{2}$, $m=\frac{2}{3}$, $p=1\frac{1}{3}$, $e=r^{\frac{7}{2}}$, $f=1$, *whence*

Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}|^{\frac{2}{3}}z^{\frac{4}{3}}\dot{z}=A$.

Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}|^{\frac{2}{3}}z^{\frac{29}{6}}\dot{z}=B$.

Fl. $\overline{r^{\frac{7}{2}}+z^{7}}|^{\frac{5}{3}}z^{\frac{4}{3}}\dot{z}=P$.

Then,

I. $\frac{7}{3}P+\frac{5}{3}\times\frac{7}{2}B$, or $\frac{7}{3}\times$ Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{5}{3}}z^{\frac{4}{3}}\dot{z}+\frac{5}{3}\times\frac{7}{2}\times$ Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{2}{3}}z^{\frac{29}{6}}\dot{z}=z^{\frac{7}{3}}.\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{5}{3}}$.

II. $P=eA+fB$, or Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{5}{3}}z^{\frac{4}{3}}\dot{z}=r^{\frac{7}{2}}\times$ Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{2}{3}}z^{\frac{4}{3}}\dot{z}+f\times$ Fl. $\overline{r^{\frac{7}{2}}+z^{\frac{7}{2}}}\big|^{\frac{2}{3}}z^{\frac{29}{6}}\dot{z}$.

Thus by making *n*, *p*, *m*, any numbers, the Propoſition may be varied infinitely, and any quantities put for *e* and *f*. I have been the longer in explaining this Propoſition, in order to be ſhorter in illuſtrating thoſe that follow.

PROP. XV.

Let $e+fz^{n}+gz^{2n}+hz^{3n}$ *&c.* $=V$, $t=$ *number of terms in* V.

And the Fluent of $V^{m}z^{p}\dot{z}=A$.

Fluent of $V^{m}z^{p+n}\dot{z}=B$.

Fluent of $V^{m}z^{p+2n}\dot{z}=C$.

Fluent of $V^{m}z^{p+3n}\dot{z}=D$ *&c. continued to t quantities.*

And the Fluent of $V^{m+1}z^{p}\dot{z}=P$. *Then will*

$$\left.\begin{array}{l}\text{I. }\overline{p+1}.eA+\overline{p+1+mn+n}.fB,\\ +\overline{p+1+2mn+2n}\,gC+\overline{p+1+3mn+3n}\,hD\\ +\text{ \&c. continued to } t \text{ terms.}\end{array}\right\}=z^{p+1}V^{m+1}$$

II. $P=eA+fB+gC+hD+$ &c. to *t* terms.

ILLUSTRATION.

EXAM. I.

Let $n=1$, $m=\frac{1}{2}$, $p=2$, $e+fz+gz^{2}=V$, $t=3$. *And*

Fl. $V^{\frac{1}{2}}z^{2}\dot{z}=A$.

Fl. $V^{\frac{1}{2}}z^{3}\dot{z}=B$.

Fl. $V^{\frac{1}{2}}z^{4}\dot{z}=C$.

And Fl. $V^{\frac{3}{2}}z^{2}\dot{z}=P$.

Then,

I. $3eA + 4\frac{1}{2}fB + 6gC = z^{p+1}V^{m+1}$, or $3e \times$ Fl. $V^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} + 4\frac{1}{2}f \times$ Fl. $V^{\frac{1}{2}}z^{3}\dot{z} + 6g \times$ Fl. $V^{\frac{1}{2}}z^{4}\dot{z} = z^{3}V^{\frac{3}{2}}$.

II. $P = eA + fB + gC$, or Fl. $V^{\frac{3}{2}}z^{2}\dot{z} = e \times$ Fl. $V^{\frac{1}{2}}z^{2}\dot{z} + f \times$ Fl. $V^{\frac{1}{2}}z^{3}\dot{z} + g \times$ Fl. $V^{\frac{1}{2}}z^{4}\dot{z}$, where e, f, g, repreſent any numbers.

EXAM. 2.

Let $n = 2$, $m = \frac{1}{3}$, $p = 5$, $e + fz^{2} + gz^{4} = V$, $t = 3$.

Fl. $V^{\frac{1}{3}}z^{5}\dot{z} = A$.

Fl. $V^{\frac{1}{3}}z^{7}\dot{z} = B$.

Fl. $V^{\frac{1}{3}}z^{9}\dot{z} = C$.

And Fl. $V^{\frac{4}{3}}z^{5}\dot{z} = P$.

Then,

I. $6eA + 8\frac{2}{3}fB + 11\frac{1}{3}gC = z^{p+1}V^{m+1}$, or $6e \times$ Fl. $V^{\frac{1}{3}}z^{5}\dot{z} + 8\frac{2}{3}f \times$ Fl. $V^{\frac{1}{3}}z^{7}\dot{z} + 11\frac{1}{3}g \times$ Fl. $V^{\frac{1}{3}}z^{9}\dot{z} = z^{6}V^{\frac{4}{3}}$.

II. $P = eA + fB + gC$, or Fl. $V^{\frac{4}{3}}z^{5}\dot{z} = e \times$ Fl. $V^{\frac{1}{3}}z^{5}\dot{z} + f \times$ Fl. $V^{\frac{1}{3}}z^{7}\dot{z} + g \times$ Fl. $V^{\frac{1}{3}}z^{9}\dot{z}$.

EXAM. 3.

Let $n = 3$, $m = \frac{5}{2}$, $p = 2$, $e + fz^{3} + gz^{6} = V$, $t = 3$. *And*

Fl. $V^{\frac{5}{2}}z^{2}\dot{z} = A$.

Fl. $V^{\frac{5}{2}}z^{5}\dot{z} = B$.

Fl. $V^{\frac{5}{2}}z^{8}\dot{z} = C$.

And Fl. $V^{\frac{7}{2}}z^{2}\dot{z} = P$.

Then,

I. $3eA + 13\frac{1}{2}fB + 24gC$, or $3e \times$ Fl. $V^{\frac{5}{2}}z^{2}\dot{z} + 13\frac{1}{2}f \times$ Fl. $V^{\frac{5}{2}}z^{5}\dot{z} + 24g \times$ Fl. $V^{\frac{5}{2}}z^{8}\dot{z} = z^{3}V^{\frac{7}{2}}$.

II. $P =$

II. $P = eA + fB + gC$, or $V^{\frac{7}{2}}z^2\dot{z} = e \times \text{Fl. } V^{\frac{5}{2}}z^2\dot{z} + f \times \text{Fl. } V^{\frac{5}{2}}z^5\dot{z} + g \times \text{Fl. } V^{\frac{5}{2}}z^8\dot{z}$.

EXAM. 4.

Let $n = 1$, $m = \frac{1}{2}$, $p = 3$, $e + fz + gz^2 + hz^3 = V$, $t = 4$.

Then

$$\text{Fl. } V^{\frac{1}{2}}z^3\dot{z} = A.$$
$$\text{Fl. } V^{\frac{1}{2}}z^4\dot{z} = B.$$
$$\text{Fl. } V^{\frac{1}{2}}z^5\dot{z} = C.$$
$$\text{Fl. } V^{\frac{1}{2}}z^6\dot{z} = D.$$
$$\text{And Fl. } V^{\frac{3}{2}}z^3\dot{z} = P.$$

Then,

I. $4eA + 5\frac{1}{2}fB + 7gC + 8\frac{1}{2}hD = z^{p+1}V^{m+1}$, or $4e \times \text{Fl. } V^{\frac{1}{2}}z^3\dot{z} + 5\frac{1}{2}f \times \text{Fl. } V^{\frac{1}{2}}z^4\dot{z} + 7g \times \text{Fl. } V^{\frac{1}{2}}z^5\dot{z} + 8\frac{1}{2}h \times \text{Fl. } V^{\frac{1}{2}}z^6\dot{z} = z^4V^{\frac{3}{2}}$.

II. $P = eA + fB + gC + hD$, or $\text{Fl. } V^{\frac{3}{2}}z^3\dot{z} = e \times \text{Fl. } V^{\frac{1}{2}}z^3\dot{z} + f \times \text{Fl. } V^{\frac{1}{2}}z^4\dot{z} + g \times \text{Fl. } V^{\frac{1}{2}}z^5\dot{z} + h \times \text{Fl. } V^{\frac{1}{2}}z^6\dot{z}$.

EXAM. 5.

Let $e = aa$, $f = a$, $g = -1$, $n = 3$, $m = -\frac{1}{2}$, $p = \frac{3}{2}$, and $V = aa + az^3 - z^6$.

Then

$$\text{Fl. } V^{-\frac{1}{2}}z^{\frac{3}{2}}\dot{z} = A.$$
$$\text{Fl. } V^{-\frac{1}{2}}z^{4\frac{1}{2}}\dot{z} = B.$$
$$\text{Fl. } V^{-\frac{1}{2}}z^{7\frac{1}{2}}\dot{z} = C.$$
$$\text{And Fl. } V^{\frac{1}{2}}z^{\frac{3}{2}}\dot{z} = P.$$

Then,

I. $\frac{5}{2}aaA + 4aB - 5\frac{1}{2}C = z^{p+1}V^{m+1}$, or $\frac{5}{2}aa \times \text{Fl. } V^{-\frac{1}{2}}z^{\frac{3}{2}}\dot{z} + 4a \times \text{Fl. } V^{-\frac{1}{2}}z^{4\frac{1}{2}}\dot{z} - 5\frac{1}{2} \times \text{Fl. } V^{-\frac{1}{2}}z^{7\frac{1}{2}}\dot{z} = z^{\frac{5}{2}}V^{\frac{1}{2}}$.

II. $P = aaA + aB - C$, or Fl. $V^{\frac{1}{2}} z^{\frac{3}{2}} \dot{z} = aa \times$ Fl. $V^{-\frac{1}{2}} z^{\frac{3}{2}} \dot{z} + a \times$ Fl. $V^{-\frac{1}{2}} z^{4\frac{1}{2}} \dot{z} -$ Fl. $V^{-\frac{1}{2}} z^{7\frac{1}{2}} \dot{z}$.

PROP. XVI.

Put $e + fz^n = V$, $k + lz^n = Y$, *and*

Fluent of $z^p \dot{z} V^m Y^q = A$.

Fluent of $z^{p+n} \dot{z} V^m Y^q = B$.

Fluent of $z^{p+2n} \dot{z} V^m Y^q = C$.

Also Fluent of $z^p \dot{z} V^{m+1} Y^q = P$.

Fluent of $z^{p+n} \dot{z} V^{m+1} Y^q = Q$.

Then it will be,

I. $\overline{p+1} . ekA + \overline{p+1+mn+n} \times fkB + \overline{p+1+qn+n} \times elB + \overline{p+1+mn+qn+2n} \times flC = z^{p+1} Y^{m+1} Y^{q+1}$.

II. $P = eA + fB$.

III. $Q = eB + fC$.

ILLUSTRATION.

EXAM. I.

Let $n = 2$, $m = \frac{1}{2}$, $q = \frac{1}{3}$, $p = 3$, $e + fz^2 = V$, $k + lz^2 = Y$, *whence*

Fl. $z^3 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} = A$.

Fl. $z^5 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} = B$.

Fl. $z^7 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} = C$.

Also Fl. $z^3 \dot{z} V^{\frac{3}{2}} Y^{\frac{1}{3}} = P$.

Fl. $z^5 \dot{z} V^{\frac{3}{2}} Y^{\frac{1}{3}} = Q$.

Then,

I. $4ekA + \overline{7fk + 6\frac{2}{3}el} \times B + 9\frac{2}{3}flC = z^{p+1} V^{m+1} Y^{q+1}$, or $4ek \times$ Fl. $z^3 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} + \overline{7fk + 6\frac{2}{3}el} \times$ Fl. $z^5 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} + 9\frac{2}{3}fl \times$ Fl. $z^7 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} = z^4 V^{\frac{3}{2}} Y^{\frac{4}{3}}$.

II. $P = eA + fB$, or Fl. $z^3 \dot{z} V^{\frac{3}{2}} Y^{\frac{1}{3}} = e \times$ Fl. $z^3 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}} + f \times$ Fl. $z^5 \dot{z} V^{\frac{1}{2}} Y^{\frac{1}{3}}$.

III. $Q = eB + fC$, or Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}} = e \times$ Fl. $z^5\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}} + f \times$ Fl. $z^7\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}$ where e, f, k, l, repreſent any numbers.

EXAM. 2.

Let $n = 1$, $m = \frac{3}{2}$, $q = \frac{1}{1}$, $p = 4$, $e + fz = V$, $k + lz = Y$, *whence*

Fl. $z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = A$.

Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = B$.

Fl. $z^6\dot{z}V^{\frac{3}{2}}V^{\frac{1}{2}} = C$.

Alſo Fl. $z^4\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = P$.

Fl. $z^5\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = Q$.

Then,

I. $5ekA + \overline{7\frac{1}{2}fk + 6\frac{1}{2}el} \times B + 9flC = z^{p+1}V^{m+1}Y^{q+1}$, or $5ek \times$ Fl. $z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + \overline{7\frac{1}{2}fk + 6\frac{1}{2}el} \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + 9fl \times$ Fl. $z^6\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = z^5V^{\frac{5}{2}}Y^{\frac{3}{2}}$.

II. $P = eA + fB$, or Fl. $z^4\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = e \times$ Fl. $z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + f \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}}$.

III. $Q = eB + fC$, or Fl. $z^5\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = e \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + f \times$ Fl. $z^6\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}}$.

PROP. XVII.

Let $e + fz^n + gz^{2n} + kz^{3n}$ *&c.* $= V$.
$k + lz^n + rz^{2n} + sz^{3n}$ *&c.* $= Y$, $t =$ *number of terms in* V, $\tau =$ *number of terms in* Y.

And the Fluent of $z^p\dot{z}V^mY^q = A$.

Fluent of $z^{p+n}\dot{z}V^mY^q = B$.

Fluent of $z^{p+2n}\dot{z}V^mY^q = C$.

Fluent of $z^{p+3n}\dot{z}V^mY^q = D$.

Alſo Fluent of $z^p\dot{z}V^{m+1}Y^q = P$.

Fluent of $z^{p+n}\dot{z}V^{m+1}Y^{q} = Q.$

Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^{q} = R.$

Then I ſay,

$$\text{I. } \overline{p+1}.ekA \left.\begin{matrix} +\overline{p+1+qn+n}\times el \\ +\overline{p+1+mn+n}\times fk \end{matrix}\right\} B.$$

$$\left.\begin{matrix} +\overline{p+1+2qn+2n}\times er \\ +\overline{p+1+mn+qn+2n}\times fl \\ +\overline{p+1+2mn+2n}\times gk \end{matrix}\right) C \left.\begin{matrix} +\overline{p+1+3qn+3n}\times es \\ +\overline{p+1+mn+2qn+3n}\times fr \\ +\overline{p+1+2mn+qn+3n}\times gl \\ +\overline{p+1+3mn+3n}\times hk \end{matrix}\right) D$$

$$= z^{p+1}V^{m+1}Y^{q+1}.$$

II. $P = eA + fB + gC + hD$ &c. to t terms.

III. $Q = eB + fC + gD + hE$ &c.

IV. $R = eC + fD + gE + hF$ &c.

V. $S = eD + fE + gF + hG$ &c.

continued to τ equations.

ILLUSTRATION.

EXAM. I.

Let $n=1$, $p=2$, $m=\frac{1}{2}$, $q=\frac{3}{2}$, $e+fz+gz^{2}=V$; $k+lz+rz^{2}=Y$, $t=3$, *whence*

Fl. $z^{2}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} = A.$

Fl. $z^{3}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} = B.$

Fl. $z^{4}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} = C.$

Alſo Fl. $z^{2}\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = P.$

Fl. $z^{3}\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = Q.$

Fl. $z^{4}\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = R.$

Then,

I. $3ekA + \overline{5\tfrac{1}{2}el + 4\tfrac{1}{2}fk} \times B + \overline{8er + 7fl + 6gk} \times C$, or $3ek \times \text{Fl. } z^{2}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} + \overline{5\tfrac{1}{2}el + 4\tfrac{1}{2}fk} \times \text{Fl. } z^{3}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} + \overline{8er + 7fl + 6gk} \times \text{Fl. } z^{4}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} = z^{3}V^{\frac{3}{2}}Y^{\frac{5}{2}}.$

II. $P = eA + fB + gC$, or $\text{Fl. } z^{2}\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = e \times \text{Fl. } z^{2}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} + f \times \text{Fl. } z^{3}\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} + g \times \text{Fl. } z^{4}\dot{z}V^{\frac{1}{2}}Y^{3}.$

III. $Q =$

III. $Q = eB + fC$, or Fl. $z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = e \times$ Fl. $z^3\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}} + f \times$ Fl. $z^4\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}}$.

IV. $R = eC$, or Fl. $z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{3}{2}} = e \times$ Fl. $z^4\dot{z}V^{\frac{1}{2}}Y^{\frac{3}{2}}$.

EXAM. 2.

Let $n = 2$, $p = 3$, $m = \frac{3}{2}$, $q = \frac{1}{2}$, $e + fz^2 + gz^4 + hz^6 = Vk + lz^2 + rz^4 + sz^6 = Y$, $t = 4$, $\tau = 4$.

Fl. $z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = A$.
Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = B$.
Fl. $z^7\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = C$.
Fl. $z^9\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = D$.
Alſo Fl. $z^3\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = P$.
Fl. $z^5\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = Q$.
Fl. $z^7\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = R$.

Then,

I. $4ekA + \overline{7el + 9fk} \times B + \overline{10er + 12fl + 14gk} \times C + \overline{13es + 15fr + 17gl + 19hk} \times D$, or $4ek \times$ Fl. $z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + \overline{7el + 9fk} \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + \overline{10er + 12fl + 14gk} \times$ Fl. $z^7\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + \overline{13es + 15fr + 17gl + 19hk} \times$ Fl. $z^9\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} = z^4V^{\frac{5}{2}}Y^{\frac{3}{2}}$.

II. $P = eA + fB + gC + hD$, or Fl. $z^3\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = e \times$ Fl. $z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + f \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + g \times$ Fl. $z^7\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + h \times$ Fl. $z^9\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}}$.

III. $Q = eB + fC + gD$, or Fl. $z^5\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = e \times$ Fl. $z^5\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + f \times$ Fl. $z^7\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + g \times$ Fl. $z^9\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}}$.

IV. $R = eC + fD$, or Fl. $z^7\dot{z}V^{\frac{5}{2}}Y^{\frac{1}{2}} = e \times$ Fl. $z^7\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}} + f \times$ Fl. $z^9\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{2}}$.

PROP. XVIII.

Let $V = e + fz^n$. $Y = k + lz^n$. $X = s + tz^n$.

And the Fluent of $z^p\dot{z}V^mY^qX^r = A$.

Fluent of $z^{p+n}\dot{z}V^mY^qX^r = B$.

Fluent of $z^{p+2n}\dot{z}V^mY^qX^r = C$.

Fluent of $z^{p+3n}\dot{z}V^mY^qX^r = D$.

Also Fluent of $z^p\dot{z}V^{m+1}Y^qX^r = P$.

Fluent of $z^{p+n}\dot{z}V^{m+1}Y^qX^r = Q$.

Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^qX^r = R$.

Then it will be,

$$\text{I. } \overline{p+1} \times eksA \left.\begin{array}{l} +\overline{p+1+mn+n} \times fks \\ +\overline{p+1+qn+n} \times els \\ +\overline{p+1+rn+n} \times ekt \end{array}\right\} B.$$

$$\left.\begin{array}{l} +\overline{p+1+mn+qn+2n} \times fls \\ +\overline{p+1+mn+rn+2n} \times fkt \\ +\overline{p+1+qn+rn+2n} \times elt \end{array}\right\} C \;\; +\overline{p+1+mn+qn+rn+3n} \times fltD = z^{p+1}V^{m+1}Y^{q+1}X^{r+1}.$$

II. $P = eA + fB$.

III. $Q = eB + fC$.

IV. $R = eC + fD$.

ILLUSTRATION.

EXAM. I.

Let $n = 1$, $p = 2$, $m = \frac{1}{2}$, $q = \frac{1}{3}$, $r = \frac{1}{4}$, $e + fz = V$, $k + lz = Y$, $s + tz = X$.

Then

Fl. $z^2\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = A$.

Fl. $z^3\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = B$.

Fl. $z^4\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = C$.

Fl. $z^5\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = D$.

Also Fl. $z^2\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = P$.

Fl. $z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = Q$.

Fl. $z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = R$.

Then,

I. $3eksA + \overline{4\frac{1}{2}fks + 4\frac{1}{3}els + 4\frac{1}{4}ekt} \times B +$
$\overline{5\frac{5}{6}fls + 5\frac{3}{4}fkt + 5\frac{7}{12}elt} \times C + 7\frac{1}{12}fltD = z^3V^{\frac{3}{2}}Y^{\frac{4}{3}}X^{\frac{5}{4}}$.

Or

$3eks \times \text{Fl. } z^2\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} + \overline{4\frac{1}{2}fks + 4\frac{1}{3}els + 4\frac{1}{4}ekt} \times \text{Fl.}$
$z^3\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} + \overline{5\frac{5}{6}fls + 5\frac{3}{4}fkt + 5\frac{7}{12}elt} \times \text{Fl. } z^4\dot{z}V^{\frac{1}{2}}$
$Y^{\frac{1}{3}}X^{\frac{1}{4}} + 7\frac{1}{12}flt \times \text{Fl. } z^5\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = z^3V^{\frac{3}{2}}Y^{\frac{4}{3}}X^{\frac{5}{4}}$.

II. $P = eA + fB$, or $\text{Fl. } z^2\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = e \times \text{Fl.}$
$z^2\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} + f \times \text{Fl. } z^3\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}}$.

III. $Q = eB + fC$, or $\text{Fl. } z^3\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = e \times \text{Fl.}$
$z^3\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} + f \times \text{Fl. } z^4\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}}$.

IV. $R = eC + fD$, or $\text{Fl. } z^4\dot{z}V^{\frac{3}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} = e \times \text{Fl.}$
$z^4\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}} + f \times \text{Fl. } z^5\dot{z}V^{\frac{1}{2}}Y^{\frac{1}{3}}X^{\frac{1}{4}}$.

Where e, f, k, l, s, t, repreſent any numbers.

EXAM. 2.

Let $n = 2$, $p = 1$, $m = \frac{1}{3}$, $q = \frac{1}{2}$, $r = 3$, $e + fz^2 = V$; $k + lz^2 = Y$, $s + tz^2 = X$.

Then

Fl. $z\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 = A$.
Fl. $z^3\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 = B$.
Fl. $z^5\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 = C$.
Fl. $z^7\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 = D$.
Alſo *Fl.* $z\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = P$.
Fl. $z^3\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = Q$.
Fl. $z^5\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = R$.

Then,

I. $2eksA + \overline{4\frac{2}{3}fks + 5els + 10ekt} \times B + \overline{7\frac{2}{3}fls +}$
$\overline{12\frac{2}{3}fkt + 13elt} \times C + 15\frac{2}{3}fltD = z^2V^{\frac{4}{3}}Y^3X^4$. Or
$2eks \times \text{Fl. } z\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + \overline{4\frac{2}{3}fks + 5els + 10ekt} \times \text{Fl.}$

$z^3\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + \overline{7\frac{2}{3}fls + 12\frac{2}{3}fkt + 13elt} \times$ Fl. $z^5\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + 15\frac{2}{3}flt \times$ Fl. $z^7\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 = zzV^{\frac{4}{3}}Y^{\frac{3}{2}}X^4$.

II. $P = eA + fB$, or Fl. $z\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = e \times$ Fl. $z\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + f \times$ Fl. $z^3\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3$.

III. $Q = eB + fC$, or Fl. $z^3\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = e \times$ Fl. $z^7\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + f \times$ Fl. $z^5\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3$.

IV. $R = eC + fD$, or Fl. $z^5\dot{z}V^{\frac{4}{3}}Y^{\frac{1}{2}}X^3 = e \times$ Fl. $z^5\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3 + f \times$ Fl. $z^7\dot{z}V^{\frac{1}{3}}Y^{\frac{1}{2}}X^3$. And ſo of others.

Thus one may proceed to more quantities, making $n = 3, 4, 5,$ &c. $p = 1, 2, 3,$ &c. $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2},$ &c. &c. which may be varied infinitely; and any quantities put for e, f, k, l, s, t; ſo general are theſe Propoſitions.

PROP. XIX.

$$\text{Let } \overline{e+fz^n}|^{m}.d\dot{z}z^{p-1} = F.$$
$$\overline{e+fz^n}|^{m}.b\dot{z}z^{p+n-1} = G.$$
$$\overline{e+fz^n}|^{m+1}.\dot{z}z^{p-1} = H.$$

Then 1. By Cor. 2. *Prop.* XIV. $pH + \overline{1+m}.\frac{nfG}{b} = z^p.\overline{e+fz^n}|^{m+1}$.

2. $H = \frac{eF}{d} + \frac{fG}{b}$, whence by common Algebra,

3. $$G = \frac{bdz^p.\overline{e+fz^n}|^{m+1} - pbeF}{pdf.+dfn.\overline{1+m}}.$$

For aſſume $H = \frac{vF}{d} + \frac{zG}{b}$, then $\dot{H} = \frac{v\dot{F}}{d} + \frac{z\dot{G}}{b}$, or $\overline{e+fz^n}|^{m+1}\dot{z}z^{p-1} = v.\overline{e+fz^n}|^{m}\dot{z}z^{p-1} + z.\overline{e+fz^n}|^{m}.\dot{z}z^{p+n-1}$, therefore by diviſion $e+fz^n.$

$z^{p-1} = vz^{p-1} + z^{p+n-1}$, or $ez^{p-1} + fz^{p+n-1} = vz^{p-1} + z^{p+n-1}$, and comparing the homologous terms, $e = v$, $f = z$, the like for any other, whence $H = \frac{eF}{d} + \frac{fG}{b}$.

PROP. XX.

The Fluent of $\overset{A}{\overline{e+fz^n}|^m a\dot{z}z^{v-1}} + \overset{B}{\overline{e+fz^n}|^m b\dot{z}z^{v+n-1}} + \overset{C}{\overline{e+fz^n}|^m c\dot{z}z^{v+2n-1}} + \overset{D}{\overline{e+fz^n}|^m d\dot{z}z^{v+3n-1}}$ *&c. will be* $A + \frac{abz^v.\overline{e+fz^n}|^{m+1} - v.ebA}{af.\overline{n+nm+v}} + \frac{cb.z^{v+n}.\overline{e+fz^n}|^{m+1} - v + n.ceB}{bf.\overline{2n+nm+v}} + \frac{cdz^{v+2n}.\overline{e+fz^n}|^{m+1} - v + 2n.deC}{cf.\overline{3n+nm+v}} + \frac{gdz^{v+3n}.\overline{e+fz^n}|^{m+1} - v + 3n.geD}{df.\overline{4n+nm+v}}$ &c. *ad infinitum*, where the law of continuation is viſible.

For,

1. $\overline{e+fz^n}a\dot{z}z^{v-1} = ac\dot{z}z^{v-1} + af\dot{z}z^{v+n-1}$, and multiplying both ſides by $\overline{e+fz^n}|^m$, then $\overline{e+fz^n}|^{m+1} a\dot{z}z^{v-1} = \overline{e+fz^n}|^m ae\dot{z}z^{v-1} + \overline{e+fz^n}|^m af\dot{z}z^{v+n-1}$; that is, putting $Q =$ the Fluent of $\overline{e+fz^n}|^{m+1} \dot{z}z^{v-1}$, $a\dot{Q} = e\dot{A} + \frac{af\dot{B}}{b}$; therefore $aQ = eA + \frac{afB}{b}$, and $Q = \frac{eA}{a} + \frac{fB}{b}$.

2. $z^v.\overline{e+fz^n}|^{m+1}$ in Fluxions is $v\dot{z}z^{v-1}.\overline{e+fz^n}|^{m+1} + \overline{1+m}.fn\dot{z}z^{v+n-1}$; and $\overline{e+fz^n}|^m = v\dot{Q} +$

$v\dot{Q} + \overline{1+m}.\,\frac{fn\dot{B}}{b}$, but the contemporary Fluents of the ſame or equal Fluxions are equal, conſequently $z^{v}.\overline{e+fz^{n}}|^{m+1} = vQ + \overline{1+m}.\,\frac{fnB}{b}$, now if Q above be here ſubſtituted, we get $z^{v}.\overline{e+fz^{n}}|^{m+1} = \frac{veA}{a} + \frac{vfB}{b}$, which by the rules of common Algebra give $B = \frac{abz^{v}.\overline{e+fz^{n}}|^{m+1} - vebA}{af.\overline{n+nm+v}}$.

After the ſame manner (by taking $\overline{e+fz^{n}}.\, b\dot{z}z^{v+n-1} = bc\dot{z}z^{v+n-1} + bf\dot{z}z^{v+2n-1}$ &c.) we get $C = \frac{cbz^{v+n}.\overline{e+fz^{n}}|^{m+1} - \overline{v+n}.ceB}{bf.\overline{2n+nm+v}}$, and $D = \frac{cdz^{v+2n}.\overline{e+fz^{n}}|^{m+1} - \overline{v+2n}.deC}{cf.\overline{3n+nm+v}}$.

It is evident from the foregoing page, that if any one of the Fluents A, B, C, &c. be given, any or all the reſt may be found, certainly the foregoing Propoſition will be of good uſe in comparing Fluents.

COR. I.

If A were given to find C, we have $B = \frac{abz^{v}\,\overline{e+fz^{n}}|^{m+1} - vebA}{af.\overline{n+nm+v}}$, and $C = \frac{cbz^{v+n}.\overline{e+fz^{n}}|^{m+1} - \overline{v+n}.ceB}{bf.\overline{2n+nm+v}}$, therefore if for B be written its value, we get C =

$$\frac{\overline{n+nm+v}.afcz^{v+n}.\overline{e+fz^{n}}|^{m+1} - \overline{v+n}.acez^{v}.\overline{e+fz^{n}}|^{m+1} + \overline{v+n}.vce^{2}A}{af^{2}.\overline{n+nm+v}.\overline{2n+nm+v}}.$$

COR.

COR. 2.

If given B to find D, we have C $=$ $\frac{cbz^{v+n}.\overline{e+fz^n}|^{m+1}-\overline{v+n}.ceB}{bf.\overline{2n+nm+v}}$, D $=$ $\frac{cdz^{v+2n}.\overline{e+fz^n}|^{m+1}-\overline{v+2n}.deC}{cf.\overline{3n+nm+v}}$, therefore by ſubſtitution as before, D $=$

$$\frac{\overline{2nm-nm+v}\,bdfz^{v+2n}.\overline{e+fz^n}|^{m+1}-\overline{v+2n}\,deb.\overline{e+fz^n}|^{m+1}+\overline{v+n}.\overline{v+2n}de^2B}{bff.\overline{2n+nm+v}.\overline{3n+nm+v}}.$$

After this manner we might go on to raiſe Corollaries at pleaſure; I ſhall exemplify the uſe of this Propoſition and its Corollaries in the following Examples.

EXAM. 1.

Given the Fluent of $\overline{p+qz^n}|^{-1}d\dot{z}z^{n-1}$, *to find the Fluent of* $\overline{p+qz^n}|^{-1}d\dot{z}z^{3n-1}$.

Here A is given to find C, $m=-1$, $p=e$, $q=f$, $v=n$, $d=a=c$, therefore by ſubſtitution at Cor. 1, C $=$

$$\frac{n^2dqz^{2n}.\overline{p+qz^n}|^{-1+1}-2n^2dpz^n.\overline{p+qz^n}|^{-1+1}+2n^2dp^2A}{dq^2n.\overline{n+n}}$$

$$=\frac{n^2dqz^{2n}-2n^2dpz^n+2n^2dp^2A}{2n^2dq^2}=\frac{dqz^{2n}-2dpz^n+2np^2A}{2nq^2}.$$

EXAM. 2.

Given the Fluent of $\overline{p+qz^n}|^{-\frac{1}{2}}l\dot{z}z^{\frac{3n}{2}-1}$ *to find that of* $\overline{p+qz^n}|^{-\frac{1}{2}}b\dot{z}z^{\frac{5n}{2}-1}$.

Here it appears A is given to find B, $m = -\frac{1}{2}$, $v = \frac{3n}{2}$, $l = b$, $p = e$, $q = f$, which being reſpectively wrote for m, v, l, &c. in the general Fluent we get $B = \dfrac{lb.z^{\frac{3n}{2}}\sqrt{p+qz^n} - \frac{3n}{2}bpA}{2lqn}$.

It will be eaſy, as occaſion ſerves from theſe and the like Fluents, to gain others more ſimple. As for inſtance;

EXAM. 3.

Given the Fluent of $\dfrac{y^2\dot{y}}{2\sqrt{bb-yy}}$, *to find the Fluent of* $\dfrac{by^4\dot{y}}{3bb\sqrt{bb-yy}}$.

Here $l = \frac{1}{2}$, $b = \dfrac{b}{3bb}$, $p = bb$, $q = -1$, $n = 2$, $y = z$; therefore the Fluent ſought becomes

$$\frac{\frac{b}{6bb}y^3\sqrt{bb-yy} - bA}{-2} = \overline{\frac{A}{2} - \frac{y^3\sqrt{bb-yy}}{12bb}} \times b.$$

I ſhall annex here a general Propoſition, which was communicated to me ſome years ago by the ingenious Mr. *Powle* of *Hereford*; it is deduced, as the above, from *Cor.* 2, *Prop.* IV. of Mr. *Emerſon*'s Fluxions, being *Cor.* 3, *Prop.* XIV. of the foregoing, drawn out into ſeveral particular Caſes, and the Fluents of each may be found by Form 11, of his Table.

PROP.

PROP. XXI.

To find the Fluent of $\overset{F}{\frac{x^{n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}} + \overset{G}{\frac{x^{2n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}} + \overset{H}{\frac{x^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}} + \overset{I}{\frac{x^{4n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}} + \overset{K}{\frac{x^{5n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}}$ *&c. ad infinitum.*

SOLUTION.

Put $A = \sqrt{fx^n+gx^{2n}}$, then $\dot{A} = \frac{fnx^{n-1}\dot{x}}{2\sqrt{fx^n+gx^{2n}}} + \frac{gnx^{2n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{fn\dot{F}}{2} + gn\dot{G}$. Therefore $\dot{G} = \frac{\dot{A}}{gn} - \frac{f\dot{F}}{2g}$, and $G = \frac{A}{gn} - \frac{fF}{2g}$.

Again, $\dot{H} = \frac{x^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}$ is $= \frac{x^{4n-1}\dot{x}}{\sqrt{fx^{3n}+gx^{4n}}}$.

Put $B = \sqrt{fx^{3n}+gx^{4n}}$, and then $\dot{B} = \frac{3fnx^{3n-1}\dot{x}}{2\sqrt{fx^{3n}+gx^{4n}}} + \frac{2gnx^{4n-1}\dot{x}}{\sqrt{fx^{3n}+gx^{4n}}}, = \frac{3fnx^{2n-1}\dot{x}}{2\sqrt{fx^n+gx^{2n}}} + \frac{2gnx^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}$ (Dividing by x^n); that is, $\dot{B} = \frac{3fn\dot{G}}{2} + 2gn\dot{H}$.

Therefore $\dot{H} = \frac{\dot{B}}{2gn} - \frac{3f\dot{G}}{4g}$; and $H = \frac{B}{2gn} - \frac{3fG}{4g}, = \frac{Ax^n}{2gn} - \frac{3fG}{4g}$; for $B = Ax^n$.

In like manner is found $I = \frac{Ax^{2n}}{3gn} - \frac{5fH}{6g}$, $K = \frac{Ax^{3n}}{4gn} - \frac{7fI}{8g}$ &c. Therefore the required Fluent will ſtand as under, where the law of continuation is evident.

$$F = F$$

$$G = \frac{A}{gn} - \frac{fF}{2g}. \text{ Here } A = \sqrt{fx^n + gx^{2n}}.$$

$$H = \frac{Ax^n}{2gn} - \frac{3fG}{4g}.$$

$$I = \frac{Ax^{2n}}{3gn} - \frac{5fH}{6g}.$$

$$K = \frac{Ax^{3n}}{4gn} - \frac{7fI}{8g}.$$

$$L = \frac{Ax^{4n}}{5gn} - \frac{9fK}{10g} \text{ \&c. \textit{ad infinitum}.}$$

OBSER. Having any one of the Fluents F, G, H, &c. any, or all the reſt, may be found. Therefore this Propoſition will be of excellent uſe in comparing Fluents.

COR. 1.

If the Fluent of $\overset{F}{\frac{\dot{x}}{\sqrt{ax + xx}}} + \overset{G}{\frac{x\dot{x}}{\sqrt{ax + xx}}} + \overset{H}{\frac{x^2\dot{x}}{\sqrt{ax + xx}}}$ &c. were required.

Here $n = 1$, $g = 1$, $f = a$, and therefore the required Fluent is

$$F = F.$$

$$G = A - \frac{aF}{2}.$$

$$H = \frac{Ax}{2} - \frac{3aG}{4}.$$

$$I = \frac{Ax^2}{3} - \frac{5aH}{6}.$$

$$K = \frac{Ax^3}{4} - \frac{7aI}{8} \text{ \&c. infinitely.}$$

COR.

COR. 2.

If the Fluent of $\overset{F}{\dfrac{x\dot{x}}{\sqrt{ax^2+cx^4}}} + \overset{G}{\dfrac{x^3\dot{x}}{\sqrt{ax^2+cx^4}}} + \overset{H}{\dfrac{x^5\dot{x}}{\sqrt{ax^2+cx^4}}}$ &c. be ſought.

Here $n = 2$, $g = c$, $f = a$, and therefore the Fluent becomes

$$F = F.$$

$$G = \frac{A}{2c} - \frac{aF}{2c}.$$

$$H = \frac{Ax^2}{4c} - \frac{3aG}{4c}.$$

$$I = \frac{Ax^4}{6c} - \frac{5aH}{6c}.$$

$$K = \frac{Ax^6}{8c} - \frac{7aI}{8c} \text{ \&c.} =$$

COR. 3.

If required the Fluent of $\overset{F}{\dfrac{x^3\dot{x}}{\sqrt{tx^4+px^8}}} + \overset{G}{\dfrac{x^7\dot{x}}{\sqrt{tx^4+px^8}}} + \overset{H}{\dfrac{x^{11}\dot{x}}{\sqrt{tx^4+px^8}}}$ &c.

Here $n = 4$, $g = p$, $f = t$, and the required Fluent

$$F = F.$$

$$G = \frac{A}{4p} - \frac{tF}{2p}.$$

$$H = \frac{Ax^4}{8p} - \frac{3tG}{4p}.$$

$$I = \frac{Ax^8}{12p} - \frac{5tH}{6p} =$$

&c.

COR.

COR. 4.

If required the Fluent of $\overset{F}{\dfrac{x^2\dot{x}}{\sqrt{x^3+px^6}}} + \dfrac{x^5\dot{x}}{\sqrt{x^3+px^6}} + \dfrac{x^8\dot{x}}{\sqrt{x^3+px^6}}$ &c.

Here $n = 3$, $g = p$, $f = 1$, and therefore the required Fluent will be

$F = F.$

$G = \dfrac{A}{3p} - \dfrac{F}{2p}.$

$H = \dfrac{Ax^3}{6p} - \dfrac{3G}{4p}.$

$I = \dfrac{Ax^6}{9p} - \dfrac{5H}{6p}.$

&c.

COR. 5.

The Fluxion $\overset{F}{\dfrac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{fx^{\frac{1}{2}}+x}}} + \overset{G}{\dfrac{\dot{x}}{\sqrt{fx^{\frac{1}{2}}+x}}} + \overset{H}{\dfrac{x^{\frac{1}{2}}\dot{x}}{\sqrt{fx^{\frac{1}{2}}+x}}}$ &c. is propoſed, to find the Fluent.

Here $n = \frac{1}{2}$, $g = 1$, $f = f$, and the Fluent

$F = F.$

$G = \dfrac{A}{\frac{1}{2}} - \dfrac{fF}{2}.$

$H = \dfrac{Ax^{\frac{1}{2}}}{1} - \dfrac{3fG}{4}.$

$I = \dfrac{Ax}{1\frac{1}{2}} - \dfrac{5fH}{6}.$

&c.

COR. 6.

Having the Fluxion $\overset{F}{\dfrac{x^{-\frac{1}{3}}\dot{x}}{\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}}} + \overset{G}{\dfrac{x^{\frac{1}{3}}\dot{x}}{\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}}}$ &c. given to find its Fluent.

$F = F$

$$F = F.$$

$$G = \frac{A}{\frac{2}{3}p} - \frac{aF}{2p}.$$

$$H = \frac{Ax^{\frac{2}{3}}}{\frac{4}{3}p} - \frac{3aG}{4p}.$$

&c.

COR. 7.

If there was given $\overset{F}{\frac{x^{-\frac{2}{3}}\dot{x}}{\sqrt{ax^{-\frac{1}{2}} + px^{-1}}}} + \overset{G}{\frac{x^{-2}\dot{x}}{\sqrt{ax^{-\frac{1}{2}} + px^{-1}}}} + \frac{x^{-\frac{5}{2}}\dot{x}}{\sqrt{ax^{-\frac{1}{2}} + px^{-1}}}$ &c.
Here $n = -\frac{1}{2}$, $g = p$, $f = a$, whence the Fluent is

$$F = F.$$

$$G = \frac{-A}{\frac{1}{2}p} - \frac{aF}{2p}.$$

$$H = -\frac{Ax^{-\frac{1}{2}}}{p} - \frac{3aG}{4p}.$$

&c.

Some Examples, ſhewing the uſe of the foregoing Propoſition and its Corollaries.

EXAM. I.

Given the Fluent of $\frac{\dot{y}}{\sqrt{c^2 + y^2}}$ *to find that of* $\frac{y^2\dot{y}}{\sqrt{c^2 + y^2}}$.

Here in other terms $\frac{\dot{y}}{\sqrt{c^2 + y^2}} = \frac{y\dot{y}}{\sqrt{c^2y^2 + y^4}}$, and $\frac{y^2\dot{y}}{\sqrt{c^2 + y^2}} = \frac{y^3\dot{y}}{\sqrt{c^2y^2 + y^4}}$, and comparing theſe with the *general Prop.* there is given F to find G; that $n = 2$, $f = c^2$, $g = 1$, $x = y$, and therefore G (=

$\left(= \frac{A}{gn} - \frac{fF}{2g}\right) = \frac{A}{2} - \frac{c^2F}{2}$. Or, becauſe A $=$ $y\sqrt{c^2+y^2}$, G $= \frac{y\sqrt{c^2+y^2}-c^2F}{2}$. If G $= y\sqrt{c^2+y^2} - c^2F$* be mult. by $\frac{t}{2c}$, we ſhall have the area of an hyperbola, whoſe tranſverſe diameter is t, conjugate c, and double ordinate y.

EXAM. 2.

Given the Fluent of $\frac{\dot{x}}{\sqrt{e+cx^2}}$ *to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{e+cx^2}}$?

Multiplying both numerator and denominator by x, $\frac{\dot{x}}{\sqrt{e+cx^2}} = \frac{x\dot{x}}{\sqrt{ex^2+cx^4}}$, and $\frac{x^2\dot{x}}{\sqrt{e+cx^2}} = \frac{x^3\dot{x}}{\sqrt{ex^2+cx^4}}$. Therefore by comparing as in the laſt Example, F (as before) is given to find G; $n = 2$, $f = e$, $g = c$; and therefore G $= \left(= \frac{A}{2c} - \frac{eF}{2c}\right) = \frac{x\sqrt{e+cx^2}-eF}{2c}$. For A $= x\sqrt{e+cx^2}$.

EXAM. 3.

Given the Fluent of $\frac{\dot{y}}{\sqrt{dd-4y^2}}$; *required that of* $\frac{y^2\dot{y}}{\sqrt{dd-4y^2}}$?

The above Fluxions may be expreſſed $\frac{y\dot{y}}{\sqrt{ddy^2-4y^4}}$ and $\frac{y^3\dot{y}}{\sqrt{ddy^2-4y^4}}$, ſo that here again (by comparing with

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* F $= 2.30258 \times$ log. $\frac{y+\sqrt{cc+y^2}}{c}$.

with the *general Prop.*) F is given to find G; $n=2$, $f=d^2$, $g=-4$, $x=y$. And therefore the Fluent $G = \left(= \frac{A}{gn} - \frac{fF}{2g}\right) = \frac{d^2F}{8} - \frac{A}{8}$; or, becauſe $A = y\sqrt{dd-4y^2}$, $G = \frac{d^2F - y\sqrt{dd-4y^2}}{8}$.

If D = degrees in the arc of a circle whoſe diameter is d, and y its ſine; then $\frac{{}^{*}.1745329Dd}{2}$ is the length of that arc, or Fluent of $\frac{d\dot{y}}{\sqrt{dd-4yy}}$. Therefore $\frac{.01745329\ \&c.\ D}{2}$ = the Fluent of $\frac{\dot{y}}{\sqrt{dd-4y^2}} = F$, and then $\frac{d^2F - y\sqrt{dd-4y^2}}{4}$ (by what has paſt before) is the Fluent of $\frac{2y^2\dot{y}}{\sqrt{dd-4y^2}}$, or the area of a circular ſemi-ſegment whoſe baſe is $2y$, and circle's diam. $= d$.

By means of the equation to the circle, &c. the above expreſſion for the ſemi-ſegment's area, becomes $.00218166\ \&c.\ D \times d^2 : -\ \frac{d^2xy}{8}$, where x and y are the ſine and coſine of half the ſegment's arc, when radius = 1. Or, becauſe $2xy$ is the ſine of double the arc, we obtain the following Rule for any circular ſegment.

Rule. *Multiply the arc of the ſegment in degrees by* .00218166 *&c. from this product take* $\frac{1}{8}$, *the ſine of the ſaid arc, and multiply the difference by the ſquare of the diameter of the circle, this laſt product is the area ſought.*

Exam. Let the diameter of a circle be 1, and the area of the ſemi-circle required.

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[*] .01745329 &c. is the length of one degree in the ſame parts as d is taken in.

Here, according to the Rule, $\overline{180^\circ \times .0021816 \text{ \&c.} - 0} : \times 1 = .39269880$ the area. As a proof of the accuracy of this Rule, $.39269880 \times 2 = .78539$, the area of the whole circle, which we know to be true from other methods.

EXAM. 4.

Given the Fluent of $\frac{y^2\dot{y}}{\sqrt{bb-yy}}$ *to find that of* $\frac{y^4\dot{y}}{\sqrt{bb-yy}}$.

In other terms $\frac{y^2\dot{y}}{\sqrt{bb-yy}} = \frac{y^3\dot{y}}{\sqrt{bby^2-y^4}}$, and $\frac{y^4\dot{y}}{\sqrt{bb-yy}} = \frac{y^5\dot{y}}{\sqrt{bbyy-y^4}}$. Compared with the Fluent, G is given to find H; $n = 2$, $f = bb$, $g = -1$, $x = y$, and therefore the Fluent H $\left(= \frac{Ax^n}{2gn} - \frac{2fG}{4g}\right) = \frac{3bbG}{4} - \frac{Ay^2}{4} =$ (becauſe $A = y\sqrt{bb-yy}$) $\frac{3bbG}{4} - \frac{y^3\sqrt{bb-yy}}{4}$.

If $S =$ the Fluent of $\frac{y^2\dot{y}}{2\sqrt{bb-yy}}$; 2S is the Fluent of $\frac{y^2\dot{y}}{\sqrt{bb-yy}}$, which put for G above, makes H, or the Fluent of $\frac{y^4\dot{y}}{\sqrt{bb-yy}} = \frac{3bbS}{2} - \frac{y^3\sqrt{bb-yy}}{4}$; therefore the Fluent of $\frac{b}{3bb} \times \frac{y^4\dot{y}}{\sqrt{bb-yy}}$ is $\overline{\frac{3bbS}{2} - \frac{y^3\sqrt{bb-yy}}{4}} \times \frac{b}{3bb}, = \overline{\frac{S}{2} - \frac{y^3\sqrt{bb-yy}}{12bb}} : \times b = \frac{S}{2} - \frac{z \times \overline{b^2 - 4z^2}\sqrt{bb-4z^2}}{6bb} \times b$; here I put $z =$ ſurd.

EXAM.

EXAM. 5.

Given the Fluent of $\frac{\dot{x}}{\sqrt{tx+x^2}}$, *to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{tx+xx}}$.

Here F is given to find H, &c. and *per* Cor. 1. $H = \frac{Ax}{2} - \frac{3tG}{4}$, and $G = A - \frac{tF}{2}$, therefore by ſubſtitution $H = \frac{Ax}{2} - \frac{3tA}{4} + \frac{3t^2F}{8}$; or ſince $A = \sqrt{tx+xx}$, $H = \overline{\frac{x}{2} - \frac{3t}{4}}\sqrt{tx+xx} + \frac{3ttF}{8}$.

EXAM. 6.

Given the Fluent of $\frac{2rx\dot{x}}{\sqrt{2rx-xx}}$, *to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{2rx-xx}}$.

By comparing with the general Prop. Here is given $R = 2rG$ to find H; $n = 1$, $f = 2r$, $g = -1$, and therefore $H = \left(= \frac{Ax^n}{2gn} - \frac{3fG}{4g} =\right) \frac{6rG}{4} - \frac{Ax}{2} =$ or by writing $\frac{R}{2a}$ for G, and $\sqrt{2rx-xx}$ for A, $H = \frac{3R}{4} - \frac{x\sqrt{2rx-xx}}{2}$.

Suppoſe $\dot{x}\sqrt{2rx-xx} = \dot{U}$ and u were required, being the area of a ſemi-circular ſegment whoſe verſed ſine $= x$, and circle's radius $= r$. $\dot{x}\sqrt{2rx-xx} = \frac{2rx\dot{x}}{\sqrt{2rx-xx}} - \frac{x^2\dot{x}}{\sqrt{2rx-xx}}$. Now by this Example, if the Fluent of $\frac{2rx\dot{x}}{\sqrt{2rx-xx}}$ be called F,

that

that of $\frac{x^2\dot x}{\sqrt{2rx-xx}}$ will be $\frac{3F}{4} - \frac{x\sqrt{2rx-xx}}{2}$, therefore the whole Fluent $U = F - \frac{3F}{4} + \frac{x\sqrt{2rx-xx}}{2} = \frac{x\sqrt{2rx-xx}}{2} + \frac{F}{4}$; or if the area of the whole ſegment be required, it is $x\sqrt{2rx-xx} + \frac{F}{2}$, But F in a ſeries is $\frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} : \times \frac{2a}{3} + \frac{2x}{5.2} + \frac{2.3x^2}{7.2.4.a} + \frac{2.3.5x^3}{9.2.4.6a^2}$ &c. Therefore the area of any circular ſegment will be $x\sqrt{ax-xx} + \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} \times : \overline{\frac{2a}{3} + \frac{2x}{5.2} + \frac{2.3.x^2}{7.2.4.a}}$ &c. where a is put for $2r$, or diameter of the circle. Which is an expreſſion for the circular ſegment, different to any thing I have yet ſeen, and converges as faſt as can be well deſired.

EXAM. 7.

Given the Fluent of $\dot x\sqrt{2rx-xx}$; *to find the Fluent of* $x\dot x\sqrt{2rx-xx}$.

By aſſuming $\sqrt{2rx-xx} = y$, we have given the Fluent of $\frac{y^2\dot y}{\sqrt{rr-yy}}$, to find the Fluent of $\frac{ry^2\dot y}{\sqrt{rr-yy}} - y^2\dot y$. If $U =$ the Fluent of $\frac{y^2\dot y}{\sqrt{rr-yy}}$, $Ur =$ Fluent of $\frac{ry^2\dot y}{\sqrt{rr-yy}}$, and therefore $Ur - \frac{y^3}{3}$ is the Fluent of $\frac{ry^2\dot y}{\sqrt{rr-yy}} - y^2\dot y$; or, becauſe $2rx - xx = yy$,

$= yy$, its Fluent is $rU - \frac{2ryx + x^2y}{3} = \frac{3rU - 2ryx + x^2y}{3}$.

EXAM. 8.

Given the Fluent $\dot{x}\sqrt{aa - xx}$, *to find that of* $x^2\dot{x}\sqrt{aa - xx}$.

By putting $\sqrt{aa - xx} = y$, the given Fluxions will be transformed into $\frac{-y^2\dot{y}}{\sqrt{aa - yy}}$, and $\frac{y^4\dot{y}}{\sqrt{aa - yy}} - \frac{a^2y^2\dot{y}}{\sqrt{aa - yy}}$. Put C = Fluent of $\frac{y^2\dot{y}}{\sqrt{aa - yy}}$, then Caa = the Fluent of $\frac{a^2y^2\dot{y}}{\sqrt{aa - yy}}$, and comparing as before taught, $\frac{3aaC}{4} - \frac{y^3\sqrt{aa - yy}}{4}$ is the Fluent of $\frac{y^4\dot{y}}{\sqrt{aa - yy}}$. Therefore the Fluent of $\frac{y^4\dot{y}}{\sqrt{aa - yy}} - \frac{a^2y^2\dot{y}}{\sqrt{aa - yy}}$ ($= x^2\dot{x}\sqrt{aa - xx}$) $= \frac{3aaC}{4} - a^2C - \frac{y^3\sqrt{aa - yy}}{4}$. But becauſe C is given negative, the ſigns of the terms where C is found muſt be changed, and (if x be reſtored) the required Fluent will be $\frac{1}{4}a^2C - \frac{1}{4}x \times \overline{a^2 - x^2}|^{\frac{3}{2}}$.

EXAM. 9.

Given the Fluent of $\dot{x}\sqrt{aa - xx}$, *to find that of* $\dot{x} \times \overline{aa - xx}|^{-\frac{1}{2}}$, *or* $\dot{x} \times \frac{1}{\sqrt{aa - xx}} = \frac{\dot{x}}{\sqrt{aa - xx}}$.

Now $\dot{x}\sqrt{aa - xx}$ (whoſe Fluent call C) may be reduced to $\frac{ax\dot{x}}{\sqrt{aa - xx}} - \frac{x^2\dot{x}}{\sqrt{aa - xx}} = \frac{a^2x\dot{x}}{\sqrt{a^2x^2 - x^4}}$ —

$-\frac{x^3\dot{x}}{\sqrt{a^2x^2-x^4}}$. If compared with the general Fluxion, a^2F is given to find G, $n=2$, $f=a^2$, $g=-1$, and therefore $G=\frac{{}^*A}{gn}-\frac{fF}{2g}$, $=$ (by writing $\frac{Q}{a^2}$ for F) $\frac{{}^*Q}{2}-\frac{x\sqrt{aa-xx}}{2}$. Hence the Fluent $C=Q-\frac{Q}{2}+\frac{x\sqrt{aa-xx}}{2}=\frac{Q}{2}+\frac{x\sqrt{aa-xx}}{2}$. And therefore by common algebra $Q=2C-x\sqrt{aa-xx}$. Therefore $F\left(=\frac{Q}{aa}\right)=\frac{2C}{aa}-\frac{x}{aa}\sqrt{aa-xx}$, the Fluent ſought.

EXAM. 10.

If given the Fluent of $-\frac{\dot{x}^{-\frac{1}{3}}\dot{x}}{\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}}$, *to find that of* $\frac{\dot{x}}{\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}}$.

By comparing with the general Fluxion, $n=\frac{2}{3}$, $f=a$, $g=p$; and F is given to find H, *per* Cor. 6, $G=\frac{A}{\frac{2}{3}p}-\frac{aF}{2p}$, and $H=\frac{Ax^{\frac{2}{3}}}{\frac{4}{3}p}-\frac{3aG}{4}$; or, if for G we write its equal, then $H=\frac{Ax^{\frac{2}{3}}}{\frac{4}{3}p}-\frac{3aA}{\frac{8}{3}p}+\frac{3aaF}{8p}$; or becauſe $A=\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}$, the required Fluent is $\frac{\overline{6x^{\frac{2}{3}}-9a}.\sqrt{ax^{\frac{2}{3}}+px^{\frac{4}{3}}}+3a^2F}{8p}=H$. Thus far Mr. *Powle.*

To the above, I will here ſubjoin two new and very general Propoſitions for finding and comparing Fluents, which will further evince, how general Mr. *Emerſon*'s 11th Form is.

PROP. XXII.

To find the Fluents of $\overset{A}{\overline{e+fz^n}|^{m-1}az^{v-1}\dot{z}}$, $\overline{e+fz^n}|^{m-1}$ $\overset{B}{b}z^{v+n-1}\dot{z}$, $\overset{C}{\overline{e+fz^n}|^{m-1}cz^{v+2n-1}\dot{z}}$, $\overline{e+fz^n}|^{m-1}$ $\overset{D}{d}z^{v+3n-1}\dot{z}$ *&c.*

By *Form* 11, if $\varphi =$ Fluent of $\overline{\alpha+\beta z^n}|^{\mu}z^{\pi}\dot{z}$, the Fluent of $\overline{\alpha+\beta z^n}|^{\mu}z^{\pi+n}\dot{z} = \frac{-\alpha.\overline{\pi+1}}{\beta.\pi+1+mn}\varphi + \frac{z^{\pi+1}\gamma}{\overline{\pi+1+mn}.\beta}$, when $\lambda = 1$, $\mu = m-1$, and $\gamma = \overline{\alpha+\beta z^n}|^{\mu+1}$. $\varphi = \text{Fl. } \overline{\alpha+\beta z^n}|^{\mu}z^{\pi}\dot{z}$.

Put $\alpha = e$, $\beta = f$, then $\gamma = \overline{e+fz^n}|^{m}$, and $\varphi = \text{Fl. } \overline{e+fz^n}|^{m-1}z^{\pi}\dot{z}$, then Fl. $\overline{e+fz^n}|^{m-1}z^{\pi+n}\dot{z} =$

$$\frac{-e.\overline{\pi+1}}{f\times\overline{\pi+1+mn}}\varphi + \frac{z^{\pi+1}\gamma}{f\times\overline{\pi+1+mn}}.$$

Or Fl. $\overline{e+fz^n}|^{m-1}z^{\pi+n}\dot{z} =$

$$\frac{z^{\pi+1}\gamma - e\times\overline{\pi+1}\times\text{Fl. }\overline{e+fz^n}|^{m-1}z^{\pi}\dot{z}}{f\times:\pi+1+mn}.$$

Let $\pi+n=v+n-1$, and $\pi+1=v$, $r=v+mn$.

Then Fl. $\overline{e+fz^n}|^{m-1}z^{v+n-1}\dot{z} =$

$$\frac{z^{v}\gamma - ev\times\text{Fl. }\overline{e+fz^n}|^{m-1}z^{v-1}\dot{z}}{fr}.$$

Let $\pi+n=v+2n-1$, and $\pi+1=v+n$.

Then Fl. $\overline{e+fz^n}|^{m-1}z^{v+2n-1}\dot{z} =$

$$\frac{z^{v+n}\gamma - e.\overline{v+n}\times \text{Fl. }\overline{e+fz^n}|^{m-1}z^{v+n-1}\dot{z}}{f\times\overline{r+n}}.$$

Let $\pi+n=v+3n-1$, and $\pi+1=v+2n$.

Then Fl. $\overline{e+fz^n}|^{m-1}z^{v+3n-1}\dot{z} =$

$$\frac{z^{v+2n}\gamma - e.\overline{v+2n}\times \text{Fl. }\overline{e+fz^n}|^{m-1}z^{v+2n-1}\dot{z}}{f\times\overline{r+2n}}.$$

Let $\pi+n=v+4n-1$, and $\pi+1=v+3n$.

Then Fl. $\overline{e+fz^n}|^{m-1}z^{v+4n-1}\dot{z} =$

$$\frac{z^{v+3n}\gamma - e.\overline{v+3n}\times \text{Fl. }\overline{e+fz^n}|^{m-1}z^{v+3n-1}\dot{z}}{f\times\overline{r+3n}}\ \text{\&c.}$$

That is,

$$A = a\times\text{Fl. }\overline{e+fz^n}|^{m-1}z^{v-1}\dot{z}.$$

$$B = \frac{bz^v\gamma - bev\times\frac{A}{a}}{fr}.$$

$$C = \frac{cz^{v+n}\gamma - ce.\overline{v+n}\times\frac{B}{b}}{f\times\overline{r+n}}.$$

$$D = \frac{dz^{v+2n}\gamma - de.\overline{v+2n}\times\frac{C}{c}}{f\times\overline{r+2n}}.$$

$$E = \frac{bz^{v+3n}\gamma - be.\overline{v+3n}\times\frac{D}{d}}{f\times\overline{r+3n}},$$

&c. (as by Mr. *Powle.*)

PROP. XXIII.

To find the Fluents of $\overset{F}{\frac{x^{n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}}$, $\overset{G}{\frac{x^{2n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}}$, $\overset{H}{\frac{x^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}}$, $\overset{I}{\frac{x^{4n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}}$, *&c.*

By *Form* 11, if $\varphi =$ Fluent of $\overline{\alpha + \beta z^n}|^{\mu} z^{\pi} \dot{z}$; the Fluent of $\overline{\alpha+\beta z^n}|^{\mu} z^{\pi+n} \dot{z} = \frac{-\alpha \times \overline{\pi+1}}{\beta \times \overline{\pi+1+n+\mu n}}$

$\varphi + \frac{z^{\pi+1} \times \overline{\alpha+\beta z^n}|^{\mu+1}}{\pi+1+n+\mu n}$, when $\lambda = 1$.

Put $\alpha = f$, $\beta = g$, $\mu = -\frac{1}{2}$, $z = x$, and $\varepsilon = \pi + 1 + n$, $\delta = \pi + 1 + \frac{3}{2}n$.

Therefore

$$\text{Fl. } \frac{x^{\pi+n}\dot{x}}{\sqrt{f+gx^n}} = \frac{x^{\pi+1} \times \sqrt{f+gx^n}}{\pi+1+\frac{1}{2}n : \times g} - \frac{\overline{\pi+1} \times f}{\pi+1+\frac{1}{2}n : \times g}$$

$$\times \text{ Fl. } \frac{x^{\pi}\dot{x}}{\sqrt{f+gx^n}}.$$

That is,

$$\text{Or Fl. } \frac{x^{\pi+\frac{3}{2}n}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{x^{\pi+1-\frac{1}{2}n}\sqrt{fx^n+gx^{2n}}}{\pi+1+\frac{1}{2}n : \times g} -$$

$$\frac{\overline{\pi+1} \times f}{\pi+1+\frac{1}{2}n : \times g} \times \text{ Fl. } \frac{x^{\pi+\frac{1}{2}n}\dot{x}}{\sqrt{fx^n+gx^{2n}}}.$$

Put $A = \sqrt{fx^n + gx^{2n}}$. Then

If $\pi + \frac{3}{2}n = 2n - 1$, then $\pi + 1 = \frac{1}{2}n$, and

$$\text{Fl. } \frac{x^{2n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{A}{ng} - \frac{f}{2g} \times \text{ Fl. } \frac{x^{n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}.$$

Again, let $\pi + \frac{3}{2}n = 3n - 1$, and $\pi + 1 = \frac{3}{2}n$, $\frac{\pi+1}{\pi+1+\frac{1}{2}n} = \frac{3}{4}$, then

$$\text{Fl. } \frac{x^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{x^n A}{2ng} - \frac{3f}{4f} \times \text{ Fl. } \frac{x^{2n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}.$$

Let $\pi + \frac{3}{2}n = 4n - 1$, then $\pi + 1 = \frac{5}{2}n$, $\frac{\overline{\pi+1}}{\pi+1+\frac{1}{2}n} = \frac{5}{6}$, then

$$\text{Fl. } \frac{x^{4n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{x^{2n}A}{3gn} - \frac{5f}{6g} \times \text{ Fl. } \frac{x^{3n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}.$$

Let $\pi + \frac{3}{2}n = 5n - 1$, $\pi + 1 = \frac{7}{2}n$, $\frac{\pi+1}{\pi+1+\frac{1}{2}n} = \frac{7}{8}$, and

$$\text{Fl.}\ \frac{x^{5n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}} = \frac{x^{3n}A}{4gn} - \frac{7f}{8f} \times \text{Fl.}\ \frac{x^{4n-1}\dot{x}}{\sqrt{fx^n+gx^{2n}}}.$$

And ſo on; that is,

$$F = F.$$

$$G = \frac{A}{gn} - \frac{fF}{2g}.$$

$$H = \frac{Ax^n}{2gn} - \frac{3fG}{4g}.$$

$$I = \frac{Ax^{2n}}{3gn} - \frac{5fH}{6g}.$$

$$K = \frac{Ax^{3n}}{4gn} - \frac{7fI}{8g}.$$

$$L = \frac{Ax^{4n}}{5gn} - \frac{9fK}{10g}.$$

&c.

So general is this Form.

PROP. XXIV.

To correct the Fluent of a given fluxionary equation.

After Fluents are obtained from their Fluxions, and there be any doubt of the truth of the concluſion; put thoſe Fluents now found into Fluxions again, and compare them with the Fluxions propoſed at firſt; for if they come out equal, the concluſion is certainly right, but if leſs, the Fluents muſt be ſo corrected, that their Fluxions may be equal to the Fluxions propoſed at firſt; alſo the Fluent may be aſſumed at pleaſure, and that aſſumption corrected, by putting the Fluxion of the aſſumed Fluent equal to the propoſed Fluxion, and comparing the homologous terms with one another.

Since no conſtant quantity has a Fluxion, therefore the Fluxion $\dot{x}$ of the Fluents x and $x \mp a$ is the ſame; (for a Fluxion may have an infinite number of Fluents) every Fluent which is obtained from the firſt Fluxion, may be increaſed or diminiſhed by ſome conſtant quantity; and every Fluent which is obtained from the ſecond Fluxion may be increaſed or diminiſhed by a quantity, whoſe ſecond Fluxion is nothing; and every Fluent which is got from the third Fluxion, may be increaſed or diminiſhed by a quantity whoſe third Fluxion is nothing, and ſo on *ad infinitum.*

When a compound Fluent, which is had from the propoſed Fluxion, includes one variable quantity only, as for inſtance, $\frac{2}{3}b + x\sqrt{bc+cx}$, which is the Fluent of $\dot{x}\sqrt{bc+cx}$ or $\overline{bc+cx}|^{\frac{1}{2}} \times \dot{x}$, it is uſually determined by *adding* or *ſubtracting* from the Fluent ſome conſtant quantity; if in the Fluent found there be put o inſtead of the variable quantity x, and if the Fluent then itſelf becomes likewiſe o, it is compleat. But if there ſhall be any remainder, as here, there remains $+ \frac{2}{3}b\sqrt{bc}$, this remainder is added to the Fluent firſt found with a contrary ſign, to have $\frac{2}{3}\overline{b + x} \times \overline{bc + cx}|^{\frac{1}{2}} - \frac{2}{3}b\sqrt{bc}$, the Fluent compleat.

The reaſon of this rule is, that the Fluent found may be ſuppoſed to exhibit the area of any curve whoſe abſciſſa is x, ſo that while x is $=$ o, the area expreſſed by the Fluent may likewiſe be o; whence if in the Fluent firſt found, inſtead of x, there be ſubſtituted o, and there be ſome remainder, that muſt be taken from the Fluent. Generally, ſome conſtant quantity to be added or ſubtracted is determined according to the nature and circumſtances of the queſtion.

Exam.

EXAM. 1.

Let $\dot{y} = \frac{-a\dot{x}}{\sqrt{cx}}$ *be given.*

The Fluent is $y = \frac{-2ax^{\frac{1}{2}}}{\sqrt{c}}$, but when $y = 0$, and $x = c$, the correct Fluent is $= 2a - \frac{2ax^{\frac{1}{2}}}{\sqrt{c}}$.

EXAM. 2.

Let $y\dot{x} = \dot{x}\sqrt{aa + 2az} - a\dot{x}$ *be given.*

By Form 3, the Fluent is $yx = \frac{\overline{aa+2ax}|^{\frac{3}{2}}}{3a} - ax$; put $y = 0$, then $x = 0$, and $\frac{aa}{3}$ is to be taken from it, therefore the Fluent corrected is $= \frac{\overline{aa+2ax}|^{\frac{3}{2}}}{3a} - ax - \frac{aa}{3}$.

EXAM. 3.

To correct the Fluent of $\dot{y} = \overline{a^m + x^m}|^n x^{m-1}\dot{x}$.

Now the Fluent is $y = \frac{\overline{a^m+x^m}|^{n+1}}{m.n+1}$; make $y = 0$, when $x = 0$, and the latter part of the equation becomes $\frac{\overline{a^m}|^{n+1}}{m.n+1} = \frac{a^{mn+m}}{m.n+1}$, therefore the Fluent corrected is $y = \frac{\overline{a^m + x^m}|^{n+1} - a^{mn+m}}{m\,n+1}$ a general Formula.

EXAM. 4.

Let $\overline{a^2+x^2}|^3 \times x\dot{x} = \dot{y}$ *be given.*

Here $m = 2$, $n = 3$, then $y = \frac{\overline{a^2+x^2}|^4 - a^8}{8}$ the correct Fluent.

EXAM.

EXAM. 5.

Let $\frac{\dot{x}}{x^{n-2}} - \frac{aa\dot{x}}{x^{n}}$, *or* $x^{2-n}\dot{x} - aax^{-n}\dot{x}$ *be given.*

The Fluent is $= \frac{x^{3-n}}{3-n} - \frac{aax^{1-n}}{1-n} + Q$, ſome conſtant quantity.

Suppoſe that $x = a$, then $Q = -\frac{a^{3-n}}{3-n} + \frac{a^{3-n}}{1-n} = \frac{2a^{3-n}}{\overline{3-n}.\overline{1-n}}$, and the correct Fluent is $= \frac{x^{3-n}}{3-n} - \frac{aax^{1-n}}{1-n} + \frac{2a^{3-n}}{\overline{3-n}.\overline{1-n}}$.

EXAM. 6.

Let $c\dot{y} = \frac{\dot{x}}{\sqrt{aa+xx}}$ *be given.*

By Caſe 4, Form 9, the Fluent is $= 2.30258 \times \log. \overline{x + \sqrt{aa+xx}}$. Now when $y = a$, $x = 0$, the equation becomes $ca = 2.30258 \log. a$. Therefore the correct Fluent is $cy - ca = 2.30258 \log. \overline{x + \sqrt{aa+xx}} - 2.30258 \log. a$; that is, $cy - ca = 2.30258 \times \log. \frac{x+\sqrt{aa+xx}}{a}$.

COROLLARIES.

1. The correct Fluent of $\frac{ab\dot{x}}{2\sqrt{aa+xx}}$ is $= \frac{1}{2}ab \times 2.30258 \times \log. \frac{x+\sqrt{aa+xx}}{a}$.

2. Of $\frac{a\dot{x}}{\sqrt{2ax+xx}}$ is $= 2.30258a \times \log. \frac{a+x+\sqrt{2ax+xx}}{a}$.

3. And ſince the correct Fluent of $\frac{\dot{x}}{\sqrt{aa+xx}}$ is $= 2.30258 \times \log. \frac{x+\sqrt{aa+xx}}{a} = A.$

Then 4. Of $\frac{x^2\dot{x}}{\sqrt{aa+xx}}$ is $= \frac{x\sqrt{aa+xx} - aaA}{2} = B.$

5. Of $\frac{x^4\dot{x}}{\sqrt{aa+xx}}$ is $= \frac{x^3\sqrt{aa+xx} - 3aaB}{4} = C.$

6. Of $\frac{x^6\dot{x}}{\sqrt{aa+xx}}$ is $= \frac{x^5\sqrt{aa+xx} - 5aaC}{6} = D.$

7. Of $\frac{x^8\dot{x}}{\sqrt{aa+xx}}$ is $= \frac{x^7\sqrt{aa+xx} - 7aaD}{8} = E.$

PROP. XXV.

To find the Logarithms.

The letter L, or log. prefixed to any number, denotes the logarithm of that number; thus L. 5, or log. 5, denotes the logarithm of 5, and L. or log. $\overline{a+1}$, the logarithm of $a+1$. Let x be the logarithm of any number $a+1$; now x the logarithm ſought may be found two different ways, the firſt may be called the *direct method*, becauſe in it is ſhewn how to deduce the logarithm from the number immediately. The other may be called the *indirect method*, becauſe therein x the required logarithm is not deduced immediately from the number $a+1$, but from the logarithm of the other number to be found by this indirect method.

The Direct Method.

From the ſuppoſition $x = L. \overline{a+1}$, this equation may be reckoned a general canon. (1) Let any equation be made between the terms any how compoſed of x and y, and combined any how with any other numbers, whether by addition, ſubtraction, multiplication, diviſion, or extraction of roots. (2) By the help of the equation thus aſſumed at pleaſure, a may be exterminated out of the general canon, and thus you will have an equation expreſſing the relation between the indeterminate quantities x and y. (3) Put this equation into Fluxions, and the Fluent thereof will give x the required logarithm of $a+1$ the propoſed number.

EXAM. 1.

Let us aſſume $a = y$, *then by the general canon, will* $x = \text{log.}\ \overline{1+y}$, *whoſe Fluxion is* $\dot{x} = \frac{\dot{y}}{1+y}$, *which being reduced by diviſion into a ſeries, gives*

$$\frac{\dot{y}}{1+y} = \dot{y} - y\dot{y} + y^2\dot{y} - y^3\dot{y} + y^4\dot{y} - y^5\dot{y} + y^6\dot{y} - y^7\dot{y} \text{ \&c.}$$

whoſe Fluent is $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7}$ *&c. the logarithm required.*

Whence it appears, that the logarithm thus found gives the quadrature of the hyperbola, and therefore this method diſcovers the relation between the hyperbolic ſpaces and the logarithms, as is well known to Geometers.

EXAM. 2.

Suppoſe $y = \frac{a}{a+2}$, *whence* $a + 1 = \frac{1+y}{1-y}$, *therefore by the general canon,* $x = L.\ \frac{1+y}{1-y}$, *in Fluxions*

$$\dot{x} = \frac{2\dot{y}}{1-yy}, = 2\dot{y} + 2y^2\dot{y} + 2y^4\dot{y} + 2y^6\dot{y} + 2y^8\dot{y} \text{ \&c.}$$

whoſe Fluent is $2y + \frac{2y^3}{3} + \frac{2y^5}{5} + \frac{2y^7}{7} + \frac{2y^9}{9}$ *&c.* $= 2 \times y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \frac{1}{9}y^9$ *&c.*

EXAM. 3.

Suppoſe $y = \frac{ra}{na+2n}$, *where* r, n, *denote any numbers taken at pleaſure, whence you will find* $a + 1 = \frac{r+ny}{r-ny}$, *and by the general canon,* $x = \log.\ \frac{r+ny}{r-ny}$, *its Fluxion is* $\dot{x} = \frac{2nr\dot{y}}{rr-n^2y^2} = \frac{2n\dot{y}}{r} + \frac{2n^3y^2\dot{y}}{r^3} + \frac{2n^5y^4\dot{y}}{r^5} + \frac{2n^7y^6\dot{y}}{r^7}$ *&c. by diviſion, whoſe Fluent is* $x = \frac{2ny}{r} + \frac{2n^3y^3}{3r^3} + \frac{2n^5y^5}{5r^5} + \frac{2n^7y^7}{7r^7}$ *&c.* $= 2 \times \frac{ny}{r} + \frac{n^3y^3}{3r^3} + \frac{n^5y^5}{5r^5} + \frac{n^7y^7}{7r^7}$ *&c.*

Lemma I.

Let z be the log. of any fraction $\frac{b}{a+1}$, x the logarithm of the denominator $a + 1$; then log. $b - z = x$.

Lemma II.

Let e be the exponent of any root of any number n whatſoever, then log. $\sqrt[e]{n} = \frac{\log.\ n}{e}$.

The Indirect Method.

Let $a + 1$ be the firſt number, whoſe logarithm x is required, and let b be the number produced from the multiplication of the numbers, whoſe greateſt is leſs than $a + 1$; and let z be the logarithm of $\frac{b}{a+1}$, that is, $z = \log.\ \frac{b}{a+1}$; this equation Mr. *Craig* calls the ſecond general canon.

Then for b let a number be taken any how compoſed of (a) and known numbers, but ſuch that the greateſt of them be leſs than $a+1$; when $\frac{b}{a+1}$ is expreſſed in this manner by a, and given or aſſumed numbers, then (as in the direct method) aſſume any equation between y and a, with any numbers, and by the help of this equation (a) may be exterminated out of the fraction; whence in the ſecond general canon, there will be no indeterminate quantity beſides z and y, and determined numbers taken at pleaſure. And the Fluxion of the canon thus expreſſed being put into an infinite ſeries, and the Fluent thereof taken, will give z the logarithm of the fraction $\frac{b}{a+1}$, and z being found, x the logarithm of $a+1$, the propoſed number is obtained; for by *Lem.* 1, $x = \log.\ \overline{a+1} = \log.\ b - z$, and log. b is known, becauſe it is produced from the multiplication of numbers whoſe greateſt is leſs than $a+1$ the propoſed number. And therefore this *indirect method* ſuppoſes the logarithms of all prime numbers leſs than the propoſed one to be given.

EXAM. 4.

Take $b = a$, *whence* $z = log.\ \frac{a}{a+1}$, *let* $y = 2a+1$ *the equation between* y *and* a, *with determinate numbers taken at pleaſure; by the help of this equation* (a) *may be exterminated out of the preceding equation, and* $z = log.\ \frac{y-1}{y+1}$, *which in Fluxions is* $\dot{z} = \frac{2\dot{y}}{yy-1} = 2\dot{y} \times \overline{yy-1}|^{-1}$, *whoſe Fluent is* $z = -2 \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \frac{1}{9}y^9$ *&c.* $= log\ \frac{b}{a+1}$.

Whence by *Lemma* 1.

$$x = \log.\, b + 2 \times \frac{1}{y} + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + \tfrac{1}{9}y^9$$

&c. $= \log.\, b - z = \log.\, \overline{a+1}$. Q. E. I.

EXAM. 5.

Suppoſe $b = \sqrt{aa+2a}$, *whence* $z = log.\, \frac{\sqrt{aa+2a}}{a+1}$ *aſſume* $y = 2a + 2$, *then* $z = log.\, y^{-1}\sqrt{yy-4}$, *in Fluxions is* $\dot{z} = 4\dot{y} \times \overline{y^3 - 4y}|^{-1}$ *or* $\frac{4\dot{y}}{y^3 - 4y}$, *and Fluent is* $z = -2 \times \frac{1}{yy} + \frac{2^2}{2y^4} + \frac{2^4}{3y^6} + \frac{2^6}{4y^8} + \frac{2^8}{5y^{10}}$ *&c.* $= log.\, \frac{b}{a+1}$, *and by Lemma* 1.

$$x = b + 2 \times \frac{1}{yy} + \frac{2^2}{2y^4} + \frac{2^4}{3y^6} + \frac{2^6}{4y^8} + \frac{2^8}{5y^{10}}$$

&c. $= \log.\, a + 1$. Q. E. I.

EXAM. 6.

Suppoſe we aſſume $b = \sqrt{aa+2a}$, *as in the foregoing Example, but now aſſume* $yy = 2aa + 4a + 1$. *Whence* $z = log.\, \frac{\sqrt{yy-1}}{\sqrt{yy+1}}$ *by the ſecond general canon, its Fluxion is* $\dot{z} = \frac{2y\dot{y}}{y^4 - 1} = 2y\dot{y} \times \overline{y^4 - 1}|^{-1}$, *and Fluent* $z = -\frac{1}{yy} - \tfrac{1}{3}y^6 - \tfrac{1}{5}y^{10} - \tfrac{1}{7}y^{14} - \tfrac{1}{9}y^{18}$ *&c.* $= log.\, \frac{b}{a+1}$.

And by *Lemma* 1.

$$x = \text{L.}\, b + \frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}}$$

&c. $= \log.\, a + 1$. Q. E. I.

In the two laſt Examples b^2 is aſſumed $= aa + 2a$, and it is evident, that $aa + 2a$ is produced from numbers whoſe greateſt is leſs than $a + 1$. For $a^2 + 2a = a \times \overline{a+2}$; but by ſuppoſition $a+1$ is the firſt number, then $a + 2$ is an even number, and $\frac{a+2}{2}$ is a whole number, whence $aa + 2a = a \times 2 \times \frac{a+2}{2}$ but any of theſe is leſs than $a+1$; and from the given logarithms of the parts a, 2, $\frac{a+2}{2}$, is had the logarithm of the product $aa+2a$ or b^2, and therefore by *Lem.* 2, the log. of the number b, namely, by dividing the logarithm of b^2 by 2.

SCHOLIUM.

It is manifeſt from both methods now explained, innumerable ſeries may be deduced, exhibiting the logarithm of any number. But though the value of b may be compoſed of a, and determinate quantities by various ways at pleaſure, and though likewiſe we may aſſume any equation between y and the terms compoſed of a and determinate numbers, yet it muſt be carefully obſerved, that ſuch values of b (in the direct method) and y (in both) be taken, which ſhall make that the ſeries exhibiting the logarithm may converge very faſt. And after the ſeries is found, experience will ſhew many rules to facilitate the calculation of the logarithm to be computed from this ſeries. As if the logarithm of any number $n + 1$ were to be found by the ſeries of the ſecond Example, I ſhould put $a + 1 = \frac{n+1}{n}$, whence $a = \frac{1}{n}$, and therefore $y = \frac{1}{1+2n} = \frac{a}{a+2}$; now if in that ſeries, this value of y be ſubſtituted, it will converge very faſt, and therefore will exhibit the logarithm of $\frac{n+1}{n}$ with leſs labour; from which, by *Lem.* 1, the logarithm ſought of the propoſed number $n+1$ is eaſily deduced. For

$$\log.\ \overline{n+1} = \log.\ \frac{n+1}{n} - \log.\ n.$$

PROP. XXVI.

From any given hyperbolical logarithm x to find the number $a+1$, anſwering thereto by infinite ſeries.

After the fluxionary equation is found from the equation by the foregoing rules, then the Fluent gives y, and y being found, a will be known, and conſequently $a+1$.

Example.

Let us take the firſt Example where $a=y$, where there is given any hyperbolical logarithm x to find the value of $1+y$, the number anſwering thereto.

Here $x = \log.\ \overline{y+1}$, its Fluxion is $\dot{x} = \frac{\dot{y}}{1+y}$, or $\dot{x} + y\dot{x} = \dot{y}$, which equation by putting $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and $\dot{y} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x} + 4Dx^3\dot{x}$ &c. becomes $\dot{x} + Ax\dot{x} + Bx^2\dot{x} + Cx^3\dot{x} + Dx^4\dot{x}$ &c. $= A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x} + 4Dx^3\dot{x}$ &c. dividing by $\dot{x}$, it will be $1 + Ax + Bx^2$ $Cx^3 + Dx^4$ &c. $= A + 2Bx + 3Cx^2 + 4Dx^4$ &c. whence by comparing the homologous terms (for their homology depend not at all on the coefficients, but altogether on the powers of the variable quantity x; the reſpective terms of the two laſt ſeries may be compared with each other) will be had $A=1$, $B=\frac{1}{2}$, $C=\frac{1}{2.3}$, $D=\frac{1}{2.3.4}$, $E=\frac{1}{2.3.4.5}$ &

&c. whence $y = x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5}$ &c. therefore $y + 1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5}$ &c. the number anſwering thereto.

PROP. XXVII.

To find a ſeries which ſhall exhibit the logarithm of the ſines.

Let radius $= 1$, and ſine of any arch $= a$, whence $\sqrt{1 - aa}$ is the coſine, whoſe logarithm required let be x, or $x = \log. \sqrt{1 - aa}$, then according to the rules of the direct method, aſſume at pleaſure any equation between y and a, as for inſtance, let $y = a$, then $x = \sqrt{1 - yy}$, whoſe Fluxion is $\dot{x} = \frac{-y\dot{y}}{\sqrt{1 - yy}}$, or $-y\dot{y} \times \overline{1 - yy}|^{-1}$, which in a ſeries is $1 + yy + y^4 + y^6 + y^8$ &c. which multiplied by $-y\dot{y}$ is $-y\dot{y} - y^3\dot{y} - y^5\dot{y} - y^7\dot{y} - y^9\dot{y}$ &c. whoſe Fluent is $x = -\frac{1}{2} \times y^2 + \frac{y^4}{2} + \frac{y^6}{3} + \frac{y^8}{4} + \frac{y^{10}}{5}$ &c.

COROLLARY.

The logarithms of the tangents are eaſily deduced from the logarithms of the ſines. For putting the reſt as in *Prob.* XXV, t becomes the tangent of the arch, whoſe ſine is a, then ſince $\frac{\text{ſine}}{\text{coſine}}$ = tangent (ſee my Gunnery) that is, $t = \frac{a}{\sqrt{1 - aa}}$, it is log. $t = $ log. $\frac{a}{\sqrt{1-aa}}$; but by *Lem.* 1, of the indirect method, log. $a -$ log. $\sqrt{1-aa} = $ log. $\frac{a}{\sqrt{1-aa}}$; therefore log. $t = $ log. $a -$ log. $\sqrt{1-aa}$; but the logarithms of the ſine and coſine, that is, log. a and log. $\sqrt{1-aa}$ are given by ſuppoſition, therefore their difference is the logarithm of the tangent required.

PROP. XXVIII.

To find a ſeries which ſhall exhibit the logarithms of the ſecants.

Let $a = $ tangent, radius $= 1$, then $\sqrt{1+aa}$ is the ſecant, whoſe logarithm put x, or $x = $ log. $\sqrt{1+aa}$; aſſume any equation between x and y, according to the preceding methods, viz. $a = y$, whence $x = $ log. $\sqrt{1+yy}$, its Fluxion is $\dot{x} = \frac{y\dot{y}}{\sqrt{1+yy}}$ or $y\dot{y} \times \overline{1+yy}|^{-1} = y\dot{y} \times 1 - yy + y^4 - y^6 + y^8$ &c. $= y\dot{y} - y^3\dot{y} + y^5\dot{y} - y^7\dot{y} + y^9\dot{y}$ &c. whoſe Fluent is $x = \frac{y^2}{2} - \frac{y^4}{4} + \frac{y^6}{6} - \frac{y^8}{8} + \frac{y^{10}}{10}$ &c.

SCHOLIUM.

After the ſame manner are found ſeries for the logarithms of the verſed ſines and chords; and moreover by *Prop.* XXIV. are found the ſine, tangent or ſecant, from the given logarithm of the ſine, tangent or ſecant.

PROP.

PROP. XXIX.

To change the logarithms found by ſeries (being the hyperbolic logarithms) into the common, or Brig's logarithms in the Tables.

Divide 1.0000000 &c. which is the common logarithm of 10, by 2.302585 &c. the hyperbolic logarithm of 10, found by the foregoing ſeries, and the quotient is 0.434294481 &c. which multiplied by the logarithm of any number found by the foregoing ſeries, will give the common logarithm of the ſame.

Thus the hyperbolic log. of 5 is 1.6094379 &c. which multiplied by 0.43429448 &c. gives 0.69897002 &c. for the common log. of 5 as in the Tables.

SECT. IV.

The Inveſtigation and Solution of ſome uſeful Problems in the Mathematics.

PROB. I.

De Maximis et Mimimis.

OR,

Of the greateſt and leaſt QUANTITIES.

DEFINITION.

WHEN any quantity is required to be the *greateſt*, it is called a *maximum*. And when it is required to be the *leaſt*, it is then called a *minimum.*

Now when any quantity is to be a *maximum* or *minimum*, it is in its own nature a determinate quantity; and therefore to determine any flowing quantity to be a *maximum* or *minimum*, is to make it a determinate or permanent one, inſtead of a flowing one; and ſince the Fluxion of a determinate quantity is = o, ſo muſt the Fluxion of the *maximum* or *minimum* be made = o, or that the negative parts thereof be equal to the affirmative; and this holds good in any fraction; for if a fraction is to be determined to be a *maximum* or *minimum*, the Fluxion of the numerator drawn into the denominator, muſt be equal to the Fluxion of the denominator drawn into the numerator. Whence this

RULE.

Find the Fluxion of the given equation, and make it $= 0$, then ſtrike out of the equation all thoſe quantities where the Fluxion is found, and thoſe that remain will give the *maximum* or *minimum* required.

Thus, from the given equation between the abſciſſa and ordinate, to find the value of the abſciſſa, to which an ordinate rightly applied is the greateſt or leaſt; put the equation into Fluxions, and the ratio of the Fluxion of the ordinate to the Fluxion of the abſciſſa muſt be made equal to *nothing*; or, which is the ſame, the Fluxion of the ordinate muſt be ſuppoſed $= 0$.

And if a variable quantity whoſe *maximum* or *minimum* is required, be not an ordinate of a curve, it may be ſuppoſed to be equal to an ordinate of any curve drawn into a given quantity; as for inſtance, if the variable quantity be $ax^2 = x^3$, where a is a given quantity, and x indeterminate. Now $ax^2 - x^3$ is to be a *maximum*, and therefore its Fluxion $2ax\dot{x} - 3x^2\dot{x} = 0$; dividing by $x\dot{x}$, it is $2a - 3x = 0$, therefore $x = \frac{2a}{3}$. Now inſtead of x be ſubſtituted its value $\frac{2a}{3}$ in the propoſed equation $ax^2 - x^3 = 0$, we get $\frac{4}{9}a^3 - \frac{8}{27}a^3 = \frac{4}{27}a^3$, for the *maximum* required.

EXAM. I.

To find the value of x when $a^3 - a^2x + x^3$ is a Minimum.

The equation in Fluxions is $-a^2\dot{x} + 3x^2\dot{x} = 0$ a *minimum*; expunge $\dot{x}$ it is $-a^2 + 3x^2 = 0$, and

Fig. $3x^2 = a^2$, therefore $x = \sqrt{\frac{aa}{3}} = a\sqrt{\frac{1}{3}}$. For x put its value, and it is $a^3 - a^3\sqrt{\frac{1}{3}} + \frac{a^3}{3}\sqrt{\frac{1}{3}}$, the *minimum*.

EXAM. 2.

17. *To divide a given line* AB *into two ſuch parts, that the ſquare of the one multiplied by that of the other, the product ſhall be the greateſt poſſible.*

Put AB $= a$, AC $= x$, then CB $= a - x$, and $\overline{a-x} \times xx = ax^2 - x^3 =$ a *maximum*; in Fluxions $2ax\dot{x} - 3x^2\dot{x} = 0$, dividing by $x\dot{x}$, we have $2a - 3x = 0$, whence $x = \frac{2a}{3}$.

COROLLARY.

Let AB $= a = 18$, then 12 and 6 are the two parts required, for $\overline{12}|^2 \times 6 = 864$, which no two other parts will anſwer the conditions required.

EXAM. 3.

To find the greateſt effect of a water-wheel, moved by the water running under it.

Put $a =$ velocity of the ſtream, $x =$ difference of the velocities of the ſtream and the wheel, then by Mr. *Emerſon*'s Mechanics, $xx =$ force of the ſtream againſt the buckets of the wheel, and $a - x =$ velocity of the ſurface, then $\overline{a-x} \times xx = ax^2 - x^3 =$ a *maximum*, in Fluxions and reduced $x = \frac{2a}{3}$, as above, and $a - x$ becomes $a - \frac{2a}{3}$ (by ſubſtituting $\frac{2a}{3}$ for x) $= \frac{3a-2a}{3} = \frac{1}{3}a$ the velocity of the ſtream; which is the greateſt effect that ſuch a water-wheel can perform.

Fig. 17.

EXAM. 4.

Given $x+y=40$, *to find* x *and* y, *when* $xy+2x^2+3y^2$ *is a minimum.*

1ſt. $xy+2x^2+3y^2$ in Fluxions $=x\dot{y}+y\dot{x}+4x\dot{x}+6y\dot{y}=0$, and $x+y$ in Fluxions is $\dot{y}=-\dot{x}$, which being put for $\dot{y}$ we have $-x\dot{x}+y\dot{x}+4x\dot{x}-6y\dot{x}=0$, expunging $\dot{x}$, it is $-x+y+4x-6y=0$, that is, $3x=5y$, and $x=\frac{5y}{3}$, and $y=\frac{3x}{5}$. Hence ariſes this general *Rule.*

$$\text{As } 5+3:40::\begin{cases}3:15=y.\\5:25=x.\end{cases}$$

EXAM. 5.

Given $\overline{1771-x}|^{8}\times x^9$ *a maximum, to find* x.

In Fluxions we have $-8\dot{x}\times\overline{1771-x}|^{7}\times x^9+9x^8\dot{x}\times\overline{1771-x}|^{8}=0$, that is, $-8x^9\dot{x}\times\overline{1771-x}|^{7}+9x^8\dot{x}\times\overline{1771-x}|^{8}=0$, dividing by $x^8\dot{x}\times\overline{1771-x}|^{8}$, we have $-8x+9\times\overline{1771-x}=0$, then $9\times\overline{1771-x}=8x$, and $17x=15939$; therefore $x=937\frac{10}{17}$.

EXAM. 6.

Required the value of x *when* $\frac{\overline{a+b}|^{x}-a^x}{c^{x-1}}$ *is a maximum.*

Let $\overline{a+b}|^{x}=z$, then x log. $\overline{a+b}=$ log. z, in Fluxions $\dot{x}$ log. $\overline{a+b}=\frac{\dot{z}}{z}$, or $\dot{z}=z\dot{x}$ log. $\overline{a+b}$, or $\overline{a+b}|^{x}\dot{x}\times$ log. $\overline{a+b}=\dot{z}$, but $\dot{z}$ is the Fluxion of z, which was put $=\overline{a+b}|^{x}$; therefore the Fluxion of $\overline{a+b}|^{x}=\overline{a+b}|^{x}\dot{x}\times$ log. $\overline{a+b}$; and proceeding in the ſame manner, $a^x=a^x\dot{x}\times$ log. a, and the Fluxion of c^{x-1} is $c^{x-1}\dot{x}\times$ log. c. Whence

$\overline{a+b}|^{x}\dot{x}$.

Fig. 17. $\frac{\overline{a+b}|^x\dot{x}.\ \log.\ \overline{a+b} - a^x\dot{x}.\ \log.\ a \times c^{x-1}}{c^{x-1}} = \frac{c^{x-1}\dot{x}.\ \log.\ c \times \overline{\overline{a+b}|^x - a^x}}{c^{x-1}}$ according to the nature of a *maximum*; that is,

$$\frac{\overline{a+b}|^x\dot{x}.\log.\ \overline{a+b} - a^x\dot{x}.\ \log.\ a\times c^{x-1} - c^{x-1}\dot{x}.\ \log.\ c\times\overline{\overline{a+b}|^x - a^x}}{c^{x-1}} = 0,$$

Dividing by c^{x-1}, we have $\overline{a+b}|^x \times \log.\ \overline{a+b} - a^x \times \log.\ a = \overline{\overline{a+b}|^x - a^x} \times \log.\ c$, or $\overline{a+b}|^x \times \overline{\log.\ c - \log.\ \overline{a+b}} = a^x \times \overline{\log.\ c - \log.\ a}$. Put $m = c - \log.\ \overline{a+b}$, $n = \log.\ c - \log.\ a$, then $\overline{a+b}|^x \times m = a^x \times n$, whence $x \log.\ \overline{a+b} + \log.\ m = x \log.\ a + \log.\ n$; or $x \log.\ \overline{a+b} - \log.\ a = \log.\ n - \log.\ m$. Therefore $x = \frac{\log.\ n - \log.\ m}{\log.\ \overline{a+b} - \log.\ a}$

$$= \frac{\log.\ \frac{n}{m}}{\log.\ \frac{a+b}{a}}.$$

EXAM. 7.

There is an ellipſis whoſe tranſverſe and conjugate diameters are 84 *and* 49 *reſpectively; quære, how ſhall I determine the point* F, *ſo that* FB *ſhall be a maximum?*

18. Put CE $= t$, BE $= c$, HF $= x$, then by the equation of the curve, we get HE $= \frac{c}{t}\sqrt{t^2-x^2}$; therefore BH $= c + \frac{c}{t}\sqrt{t^2-x^2}$; and (by 47, *Ex.* 1.) $\overline{FB}|^2 = \overline{c + \frac{c}{t}\sqrt{t^2-x^2}}|^2 + x^2$, therefore

$FB = \sqrt{\overline{c + \frac{c}{t}\sqrt{t^2-x^2}}\,\Big|^2 + x^2}$; which by the Fig. 18. queſtion muſt be a *maximum*, in Fluxions we have $\overline{c + \frac{c}{t}\sqrt{t^2-x^2}} \times \frac{\overline{-cx\dot{x}}}{t\sqrt{t^2-x^2}} + x\dot{x} = 0$, which reduced gives $\overline{t^2-c^2} \times \overline{t^2-x^2}\big|^{\frac{1}{2}} = tc^2$; put $m = t^2 - c^2$, then will $x = \sqrt{t^2 - \frac{t^2c^4}{m^2}}$ which will be a canon for finding HF = 36 *ferè*, therefore $HB = c + \frac{c^3}{m}$, and $BF = \sqrt{\overline{c + \frac{c^3}{m}}\,\Big|^2 + x^2}$ the *maximum*. Q. E. I.

Exam. 8.

A beam BC *is to be ſupported in a given poſition by means of a prop* DE, *of a given length, inſiſting on the horizontal beam* AB; *it is propoſed to determine the poſition of the prop, ſo that the beam* AB *whereon it ſtands may be the leaſt ſubject to break, or ſo that the force, whereby it actually tends to break, may be to the whole force it can ſuſtain at* E *in the leaſt ratio poſſible, the thickneſs of the beam being every where the ſame.* 19.

Let $AB = b$, $ED = a$, G the center of gravity and IG perpendicular to BG, $BG = d$, $BD = v$, s, $B = s$, coſ. $= c$, s, $D = x$, s, $E = y$; by Trig. $\frac{y}{v} = \frac{s}{a}$, and $EB = \frac{ax}{s}$, w = weight of BC, the relative weight of the beam down the inclined plane GI is $= cw$ (Prop. XXXI. of Mr. *Emerſon*'s Mechanics). The ſolution of the queſtion depends upon this, that action and re-action being equal, as much as the beam DC thruſts DE, ſo much does DE thruſt or act againſt AB in direction ED. Let F = that thruſt, then the effect of that force at D being

Fig. being as the ſine of the $\angle D$; we ſhall have by
19. the power of the leaver BDG, $BG \times cw = BD \times Fx$, or $dcw = vFx$ (Mechan. Prop. XIX.) and $F = \frac{dcw}{vx}$. Again (by Prop. LXX. *Mechanics*) the ſtreſs at $E = Fy \times AEB = \frac{dcwy}{vx} \times \overline{b - \frac{ax}{s}} \times \frac{ax}{s} = \frac{dcws}{ax} \times \frac{ax}{s} \times \frac{bs - ax}{s} = dcw \times \frac{bs - ax}{s} = \frac{dcwa}{s} \times \overline{\frac{bs}{a} - x} = minimum$ per queſtion, therefore $\frac{bs}{a} - x = min.$ and therefore $x = max.$ that is, $x = 1$, and the angle BDE a right angle. Q. E. I.

EXAM. 9.

To find the greateſt cylinder that can be inſcribed in a given cone.

20. Put the altitude of the cone $CD = a$; $BD = DA = b$; $c = 3.1416$; $GD = x$; $EG = GF = y$, then $CG = a - x$. By the ſimilar triangles CDB, CGF, it will be $CG : GF :: CD : DB$, that is, $a - x : y :: a : b$, whence $ab - bx = ay$, and $x = \frac{ab - ay}{b}$. But $cyyx$ = ſolidity of the cylinder = a *maximum*, in which expreſſion, for x ſubſtitute its equal, then $cyy \times \frac{ab - ay}{b} = \frac{abcyy - acy^3}{b} = \frac{ac}{b} \times \overline{by^2 - y^3}$ = a *maximum*, and $by^2 - y^3 = maximum$, in Fluxions $2by\dot{y} - 3y^2\dot{y} = 0$; dividing by $y\dot{y}$, then $2b = 3y$ and $y = \frac{2b}{3}$, for y ſubſtitute $\frac{2b}{3}$ its equal in $\frac{ab - ay}{b}$, then $x = \frac{ab - \frac{2ab}{3}}{b} = \frac{3ab - 2ab}{3b} =$

$\frac{3a-2a}{3} = \frac{1}{3}a$. The length of the cylinder muſt be $\frac{1}{3}$ of the cone's altitude. Fig. 20.

EXAM. 10.

Given the ſolidity of a ſquare pyramid, to find that which has the leaſt ſurface, excluſive of the baſe.

Put $a =$ ſolidity, $x =$ height, $y =$ breadth, then the ſlant ſide is $= \sqrt{xx + \frac{yy}{4}}$, by Theo. 11, of my *Gunnery*, and $2y\sqrt{xx + \frac{yy}{4}} =$ ſurface, but $\frac{x}{3} \times yy = a$, then $yy = \frac{3a}{x}$; for yy ſubſtitute $\frac{3a}{x}$ its equal in $2y\sqrt{xx + \frac{yy}{4}}$, then the ſurface is $= 2\sqrt{\frac{3a}{x}}\sqrt{xx + \frac{3a}{4x}} = 2\sqrt{3ax + \frac{9aa}{4xx}} = 3ax + \frac{9aa}{4xx} =$ a *maximum*; in Fluxions $3a\dot{x} - \frac{72aax\dot{x}}{16x^4} = 0$, dividing by $3a\dot{x}$, then $1 - \frac{24ax}{16x^4} = 0$, and $16x^4 = 24ax$, whence $x^3 = \frac{3a}{2}$ and $x = \sqrt[3]{\frac{3a}{2}}$, and conſequently $y = \sqrt{\frac{3a}{x}}$.

EXAM. 11.

To find the ſlant ſide of a ſquare pyramid the leaſt poſſible, when the ſolidity is given.

Let $a =$ ſolidity, $x =$ perpendicular altitude, $y =$ breadth, then by Theo. 11, Geo. the ſlant ſide is $= \sqrt{xx + \frac{yy}{4}}$, but $\frac{x}{3} \times yy = a$, then $yy = \frac{3a}{x}$, for

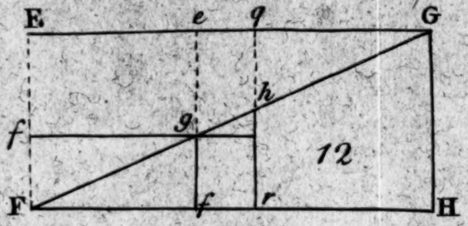
Fig. 20. for yy ſubſtitute its equal, then $\sqrt{xx + \frac{yy}{4}}$ becomes $\sqrt{xx + \frac{3a}{4x}}$ = a *minimum*; and $xx + \frac{3a}{4x}$ = *minimum*, in Fluxions $2x\dot{x} - \frac{12a\dot{x}}{16x^2} = 0$, divide by $2\dot{x}$, it is $x - \frac{6a}{16x^2}$, then $16x^3 = 6a$ and $x = \sqrt[3]{\frac{3a}{8}}$.

EXAM. 12.

Given the ſlant ſide of a fruſtum of a cone = 60 *inches, and the diameters in proportion as* 2 *to* 3, *required the ſolidity when a maximum.*

21. Here is given AB = 60 = m; BC : AE : : 2 : 3 : : 1 : $\frac{3}{2} = n$, let x = BC, then nx = AE, hence AD = $\frac{n-1}{2}x$, and BD = $\sqrt{m^2-a^2x^2}$, where $a = \frac{n-1}{2}$, the ſolidity = $\overline{1+n+n^2} \times \frac{.7854}{3} \times xx\sqrt{m^2-a^2x^2}$ = a *max.* or = $m^2x^4 - a^2x^6$ a *max.* in Fluxions $4m^2x^3\dot{x} - 6a^2x^5\dot{x} = 0$, dividing by $x^3\dot{x}$, it is $4m^2 - 6a^2x^2 = 0$, whence $x^2 = \frac{4m^2}{6a^2}$ and $x = \frac{2m}{a\sqrt{6}}$ = 195.9592 = BC, hence AE = $\frac{2mn}{a\sqrt{6}}$ = 293.9388 and BD = $\frac{m}{\sqrt{3}}$ = 34.641016, and the ſolidity = $\frac{\overline{1+n+nn} \times .4030665m^3}{\overline{n-1}|^2}$ = 1654184.916 cubic inches.

EXAM.


12
E
e
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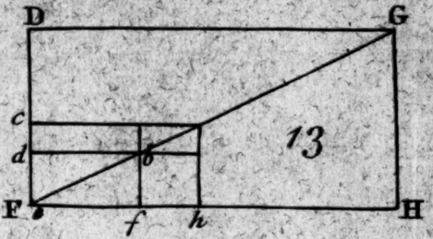
f
g
F
f
r
H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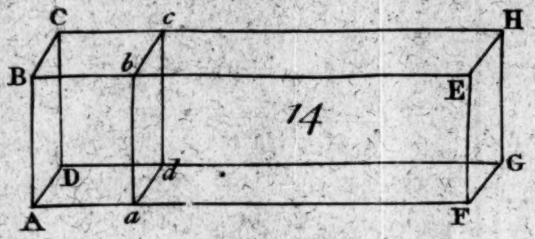
h
H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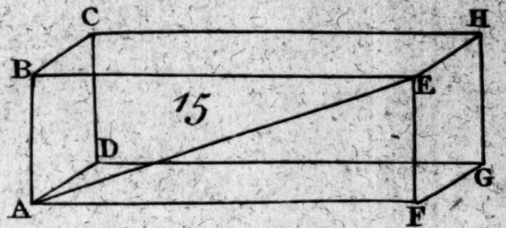
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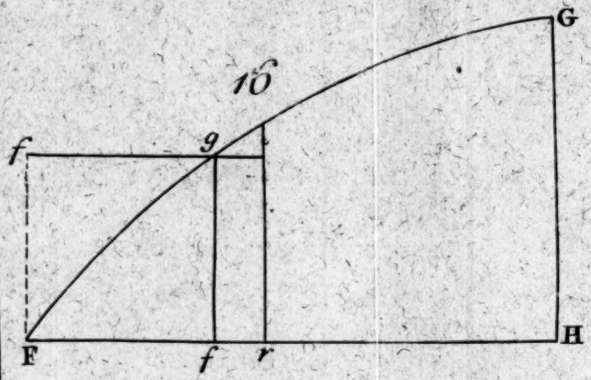
16
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r
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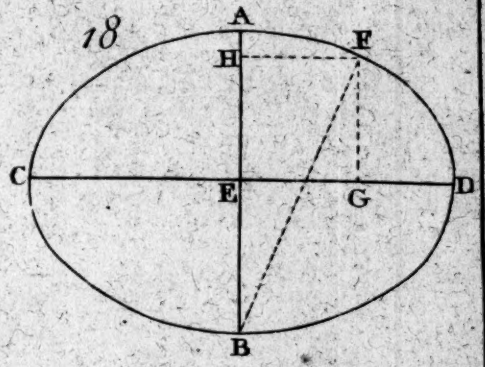
18
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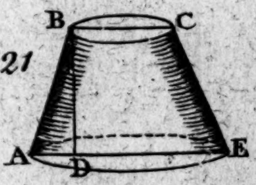
21
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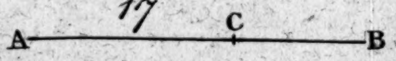
17
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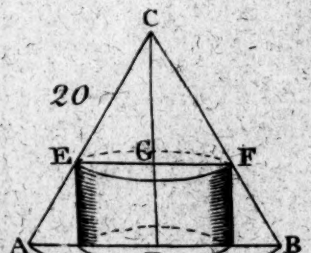
20
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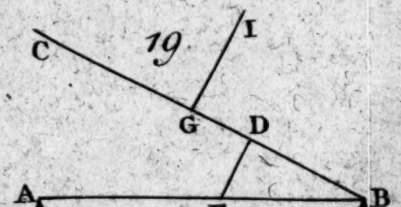
19
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Fig. 21.

Exam. 13.

Given $dx^{m^{\frac{2}{3}}}+by^n+cz^{a^{\frac{3}{5}}}=g$, *find ſuch values of* x, y, *and* z, *as ſhall make* $\overline{xyz}\big|^{s}$ *a maximum.*

Put $s=m^{\frac{2}{3}}$, $t=a^{\frac{3}{5}}$, then $dx^s+by^n+cz^t=g$; and xyz per queſtion muſt be a MAXIMUM, therefore $x=\overline{\frac{g-by^n-cz^t}{d}}\Big|^{\frac{1}{s}}$, and by ſubſtitution $yz\times\overline{g-by^n-cz^t}\Big|^{\frac{1}{s}}$; or, $gz^sy^s-bz^sy^{n+s}-cz^{\overline{s+t}}y^s$ a *max.* make z flow, then will $sg-bsy^n-c\times\overline{s+t}z^t=0$; therefore $y^n=sg-c\times\overline{s+t}z^t$; make y flow, then will $sg-b\times\overline{n+s}y^n-csz^t=0$; therefore $y^n=\frac{sg-csz^t}{b\times\overline{n+s}}$; conſequently $\frac{sg-csz^t}{b\times\overline{n+s}}=\frac{sg-c\times\overline{s+t}z^t}{bs}$, therefore $z^t=\frac{gsn}{cn\times\overline{s+t}+cst}$; $y^n=\frac{gst}{bn\times\overline{s+t}+bst}$; and $x^s=\frac{gnt}{dn\times\overline{s+t}+dst}$; therefore $z=\overline{\frac{gnm^{\frac{2}{3}}}{cn\times\overline{m^{\frac{2}{3}}+a^{\frac{3}{5}}}+cm^{\frac{2}{3}}a^{\frac{3}{5}}}}\Bigg|^{\frac{1}{a^{\frac{3}{5}}}}$; $y=\overline{\frac{gm^{\frac{2}{3}}a^{\frac{3}{5}}}{bn\times\overline{m^{\frac{2}{3}}+a^{\frac{3}{5}}}+bm^{\frac{2}{3}}a^{\frac{3}{5}}}}\Bigg|^{\frac{1}{n}}$; and $x=\overline{\frac{gna^{\frac{3}{5}}}{dn\times\overline{m^{\frac{2}{3}}+a^{\frac{3}{5}}}+dm^{\frac{2}{3}}a^{\frac{3}{5}}}}\Bigg|^{\frac{1}{m^{\frac{2}{3}}}}$; the required values of x, y, and z.

Scholium.

It may be obſerved, when any quantity is a *maximum* or *minimum*, its ſquare, or its cube, &c. or its root will likewiſe be a *maximum* or *minimum*; or if any given quantity be added to it or ſubtracted from it, or if it be multiplied or divided by any given quantity, it will ſtill remain a *maximum* or *minimum*.

Exam.

EXAM. 14.

Given $\left.\begin{array}{r} x+y=a \\ x+y+z=b \end{array}\right\}$ *find ſuch values of* x, y, *and* z, *as ſhall make* zy^2z^3 *a maximum.*

Firſt to find xy^2 a *maximum.*

Here $x+y=a$, then $x=a-y$, for x ſubſtitute $a-y$ its value, then xy^2 becomes $\overline{a-y}\times yy = ay^2-y^3=$ *maximum*, which in Fluxions is $2ay\dot{y}-3y^2\dot{y}=0$, divide by $y\dot{y}$, it is $2a=3y$, and therefore $y=\frac{2a}{3}$, but $x=a-y=a-\frac{2a}{3}=\frac{3a-2a}{3}=\frac{a}{3}$, therefore $y=2x$.

2. To find xy^2z^3 a *maximum.*

Again $x+y+z=b$ by the queſtion, then $x+y=b-z$, but $y=2x$ as above, then $yy=4xx$, therefore $xy^2=4x^3$, then $4x^3z^3=max.$ but $x+y=b-z$, or $3x=b-z$, therefore $4x^3z^3=\frac{4}{3}\times\overline{b-z}|^3\times z^3=max.$ and $\overline{b-z}|^3\times z^3=max.$ or $\overline{b-z}\times z=max.$ or $bz-zz=max.$ by the foregoing Scholium, which in Fluxions is $b\dot{z}-2z\dot{z}=0$, divide by $\dot{z}$, it is $b-2z=0$, whence $2z=b$; but $b=x+y+z$, and $z=x+y=3x$ (for $y=2x$) conſequently $x+2x+3x=6x=b$, whence $x=\frac{b}{6}$; and ſince $y=2x$, then $y=\frac{b}{3}$; alſo $z=3x$, then $z=\frac{3b}{6}=\frac{b}{2}$.

EXAM. 15.

Given the hypotheneuſe of any right angled triangle; to find the baſe and perpendicular, when the area is the greateſt poſſible.

Put the hypotheneuſe $=a$, baſe $=y$, then $\sqrt{aa-yy}=$ perpendicular and $\frac{y}{2}\sqrt{aa-yy}=$

area, a *maximum*, or $\frac{aayy - y^4}{4}$, in Fluxions is $2a^2y\dot{y} - 4y^3\dot{y} = 0$, whence $y = a\sqrt{\frac{1}{2}}$ = baſe, and alſo equal to the perpendicular.

EXAM. 16.

There is a cylindric veſſel whoſe diagonal is 50 inches, required the diameter and depth, ſo as it muſt hold the moſt liquor poſſible.

Let $d = 50$ the diagonal, $n = .7854$, depth $= x$, then $\sqrt{dd - xx}$ = diameter, but the ſquare of the diameter multiplied by .7854, and that product again by the depth gives the ſolidity, that is, $\overline{dd - xx} \times n \times x = ddnx - nx^3$ the ſolidity, a *maximum*; in Fluxions $ddn\dot{x} - 3nx^2\dot{x} = 0$, divide by $n\dot{x}$, gives $dd = 3x^2$, whence $x = d\sqrt{\frac{1}{3}} = 30$, and diameter $= 40$.

A PROPOSITION

To ſolve all manner of ſimple fluxionary exponential Problems.

It is to be obſerved, that $x^{\overset{x}{x}}$ muſt be noted thus $\bar{x}|^{xx}$ (ſee the Preface) to obtain the required *minimum*.

RULE.

Divide -1 by the given exponential's numerical index, and that quotient ſet down vulgar fractionwiſe, will be the hyperbolical logarithm of x, when the given exponential is a *minimum*; and, by *Prop.* XXVI. the number anſwering to it is had by this ſeries $1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \frac{x^6}{2.3.4.5.6}$ &c. by ſubſtituting the ſaid hyperbolical logarithm of the number ſought for x, and its ſeveral powers, divided as in the ſeries, and added to unity. That is,

$$\text{When}\left\{\begin{array}{ll} x^{x} & \\ x^{x^{x}} & \text{or } \overline{x}|^{xx} \\ x^{x^{x^{x}}} & \text{or } \overline{x}|^{x^3} \\ x^{x^{x^{x^{x}}}} & \text{or } \overline{x}|^{x^4} \end{array}\right\}\begin{array}{l}\text{is a \textit{minimum}, the} \\ \text{the hyp. log. of} \\ x \text{ is}\end{array}\left\{\begin{array}{l} \frac{-1}{1} = -1 \\ \frac{-1}{2} = -\frac{1}{2} \\ \frac{-1}{3} = -\frac{1}{3} \\ \frac{-1}{4} = -\frac{1}{4} \end{array}\right.$$

Or the ſame may eaſily be inveſtigated thus.

EXAM. I.

Given $x^x = y$ a minimum, required the value of x.

Since $x^x = y$ is to be a *minimum*, it is evident that its hyperbolic logarithm, which offers itſelf firſt, will be a *minimum*, therefore $x^x = y$ in logarithms is x log. $x =$ log. y; whoſe Fluxion is $\dot{x}$ log. $x + \dot{x} = 0$, dividing by $\dot{x}$ is log $x + 1 = 0$, whence log. $x = -1$ as above, which call y, then will the number anſwering to it be had by the above general ſeries, viz. $1 + y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \frac{y^5}{2.3.4.5}$ &c. Now ſince the hyperbolic logarithm of the number here ſought is -1; let -1 be ſubſtituted for y in the above ſeries, and then it will become $+\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6}$ &c. that is,

$+\frac{1}{2}$

$$\begin{array}{llll}
+\frac{1}{2} & = +.5 & & \\
-\frac{1}{2.3} & = & & -.1666666 \\
+\frac{1}{2.3.4} & = +.0416666 & & \\
-\frac{1}{2.3.4.5} & = & & -.0083333 \\
+\frac{1}{2.3.4.5.6} & = .0013888 & & \\
-\frac{1}{2.3.4.5.6.7} & = & & -.0001984 \\
\cline{4-4}
 & & & -.1751983 \\
+\frac{1}{2.3.4.5.6.7.8} & = .0000248 & & \\
\cline{2-2}
 & +.5430802 & & \\
 & -.1751983 & & \\
\cline{2-2}
 & .3678819 = x. & &
\end{array}$$

EXAM. 2.

Given $x^{x^x} = y$, *required the value of* x *when* y *is a minimum.*

Since $\overline{x}|^{xx} = y$ is to be a *minimum*, its logarithm will be xx log. x = log. y, which in Fluxions is $2x\dot{x}$ log. $x + \frac{\dot{x}}{x} \times x^2 = 0$, that is $2x\dot{x}$ log. $x + x\dot{x} = 0$, dividing by $x\dot{x}$ is 2 log. $x + 1 = 0$, whence log. $x = -\frac{1}{2}$, the hyperbolic logarithm of x as above, which call y, then will the number anſwering to it be had (Prop. XXVI.) by this general ſeries as by the rule above, viz. $1 + y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \frac{y^5}{2.3.4.5}$ &c. Let $-\frac{1}{2}$ be ſubſtituted inſtead of y in the above ſeries, and then we ſhall

$$\begin{array}{rcll}
+1 & = & +1.00000000 & \\
-y & = & \text{———} & -.5 \\
+\frac{y^2}{2} & = & +.125 & \\
-\frac{y^3}{2.3} & = & \text{———} & -.02083333 \\
+\frac{y^4}{2.3.4} & = & +.00260416 & \\
-\frac{y^5}{2.3.4.5} & = & \text{———} & -.00026041 \\
+\frac{y^6}{2.3.4.5.6} & = & +.00002170 & \\
-\frac{y^7}{2.3.4.5.6.7} & = & \text{———} & -.00000155 \\
 & & & \overline{-.52109529} \\
+\frac{y^8}{2.3.4.5.6.7.8} & = & +.00000009 & \\
 & & \overline{+1.12762596} & \\
 & & -.52109529 & \\
 & & \overline{} &
\end{array}$$

Therefore $x = .60653067$ correct to a dec.

To find the logarithm of $.60653067$; now $1 - x$ or $1 - .6065306 = .3934694$, whence $x = -.3934694$, consequently the log. of $1 - x$ will, by *Prop.* XXV. be had by this ſeries $= x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6}$ &c. $= -.5$ very nearly.

Therefore $\overline{x}|^{x}$ or $\overline{x}|^{xx}$ is $= .832$ the *minimum* required.

EXAM. 3.

Let $\overline{x}|^{xxx} = y$, *to find* x *when* y *is a minimum.*

Since $\overline{x}|^{x^3} = y$ is to be a *minimum*, in Logarithms it will be x^3 log. $x =$ log. y, whoſe Fluxion is $3x^2\dot{x}$ log. $x + \frac{\dot{x}}{x} \times x^3$ or $3x^2\dot{x}$ log. $x + x^2\dot{x} = 0$, dividing by $x^2\dot{x}$ we get 3 log. $x + 1 = 0$, whence log. $x = -\frac{1}{3}$ the hyp. log. of x as above by the rule, which call y, and $-\frac{1}{3}$ be ſubſtituted in the above ſeries inſtead of y, you will find the value of x as in the laſt.

EXAM. 4.

Let $\overline{x}|^{x^{x^{x}}}$ *to find the value of* x *when* y *is a minimum.*

Since $\overline{x}|^{x^4} = y$ is to be a *minimum*, it will be in logarithms x^4 log. $x =$ log. y, whoſe Fluxion is $4x^3\dot{x}$ log. $x + \frac{\dot{x}}{x} \times x^4$ or $4x^3\dot{x}$ log. $x + x^3\dot{x} = 0$, dividing by $x^3\dot{x}$ it is 4 log. $x + 1 = 0$, whence log. $x = -\frac{1}{4}$ the hyp. log. of x as above, which call y, and ſubſtitute $-\frac{1}{4}$ for y in the above ſeries, you will get the value of x.

EXAM. 5.

Required the value of x *when* $x^{\frac{1}{x}}$ *is a maximum.*

The logarithm of $x^{\frac{1}{x}}$ is $\frac{1}{x}$ log. $x = 0$, in Fluxions is $\frac{-\dot{x}}{xx} - \frac{\dot{x}}{xx} \times$ log. $x = 0$, dividing by $\frac{\dot{x}}{xx}$ is $1 -$ log. $x = 0$, whence log. $x = 1$, the hyp. log. of x, and the number anſwering to it is by the rule $1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5}$ &c. and for x ſubſtitute 1, and the ſeries becomes $x = 1 +$

$1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6}$ &c. or $x = 2.7182$,

Or thus,

Since it has been found above log. $x = 1$, ſay,
hyp. log. tab. log. hyp. log. tab. log.
As $2.3025851 : 1.0000000 :: 1 : .43429$ in Brigg's Form, and the number anſwering to it is 2.7182 in the Tables.

EXAM. 6.

Required the value of x, when $\overline{x^{\frac{1}{3}}}|^{x^{\frac{2}{3}}}$ is a minimum.

Let $\overline{x^{\frac{1}{3}}}|^{x^{\frac{2}{3}}} = v$, which in logarithms is $\frac{1}{3}x^{\frac{2}{3}}$ log. $x =$ log. v, in Fluxions is $\frac{2}{9}x^{-\frac{1}{3}}\dot{x}$ log. $x + \frac{1}{3}x^{-\frac{1}{3}}\dot{x} = \frac{\dot{v}}{v}$, which multiply by v gives $\frac{2}{9}vx^{-\frac{1}{3}}\dot{x}$ log. $x + \frac{1}{3}vx^{-\frac{1}{3}}\dot{x} = v = 0$, dividing by $vx^{-\frac{1}{3}}\dot{x}$, then we get $\frac{2}{9}$ log. $x + \frac{1}{3} = 0$, then $\frac{2}{9}$ log. $x = \frac{-1}{3}$, whence log. $x = \frac{-9}{6} = -1.5$, wherefore putting $1 + y = x$ as above, we ſhould have $y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}$ &c. $= \frac{-9}{6}$, by the reverſion of which y, and conſequently x may be found equal to a ſeries of fractions as above; but for variety's ſake I chuſe rather the following method, *viz.* ſince $\frac{-9}{6} = -1.5$ is the hyperbolical logarithm of x, multiply it by .43429448 &c. and then it gives the *Briggian* logarithm of $x = -.6514417$; the Arith. Comp. of which .3485583 (being the log. of 10 times x)

gives by the common Tables $x = .2231698$ &c. the value required.

For a proof we may proceed thus;

$-\ .6514417 = \log. x$

$-\ .2171472 = \log. x^{\frac{1}{3}}$

$-\ .4342944 = \log. x^{\frac{2}{3}}$

Arith. Com. $.5657056$ whence $x^{\frac{2}{3}} = .3679795.$

$-\ .2171472$

Mult. by $.3679795$

Product $-\ .0799057$

Arith. Com. $.9200943$ hence $\overline{x^{\frac{1}{3}}}|^{x^{\frac{2}{3}}} = .8319444$

&c. the *minimum*.

If we take a number either greater or leſs than $.2231698$ &c. for x, and proceed in like manner, the product will be found greater than $.8319444$ &c. and conſequently the *minimum* is truly found.

EXAM. 7.

Required the value of x, when $\overline{x^{\frac{1}{3}}}|^{\overline{x^{\frac{5}{7}}}|^{x^{\frac{6}{13}}}}$ is a minimum.

The notation of the given exponential, by the obſervation in the Preface, is univerſally $\overline{x^n}|^{x^m} = \overline{x}|^{nx^m}$ or $\overline{x^{\frac{1}{3}}}|^{x^{\frac{30}{91}}}$; let X denote the hyp. log. of x, then $\overline{x}|^{x^m}$ in Fluxions is $\dot{X}x^m + mX\dot{x}x^{m-1} = 0$, but $\dot{X} = \frac{\dot{x}}{x}$ by the nature of logarithms, which ſubſtitute for $\dot{X}$ in the equation is $\frac{\dot{x}}{x}x^m + mX\dot{x}x^{m-1}$

Fig. $= 0$, and by diviſion it becomes $1 + mX = 0$, hence $X = \frac{-1}{m} = \frac{-1}{\frac{30}{91}} = -3\frac{1}{30}$, the hyp. log. of x, as by the rule, which call y, and $-3\frac{1}{30}$ ſubſtituted for y in the general ſeries $1 + y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \frac{y^5}{2.3.4.5}$ &c. and divided by the ſeveral powers, as has already been ſhewn, gives $x = .048153$ the value required.

PROB. II.

To draw tangents to curves, where there is given the relation of the abſciſſa and ordinate.

22. Here AP is the abſciſſa, PM the ordinate, TM the tangent, PT the ſubtangent, BM the normal or perpendicular to the tangent TM, PB the ſub-normal or ſub-perpendicular, which is ſub-duple the parameter in the parabola, and therefore an invariable quantity.

Then in the parabola AM. As PB : PM :: PM : PT.
And PB : PM :: MB : TM.

Let the abſciſſa AP $= x$, ordinate PM $= y$, tangent TM $= t$, the ſub-tangent PT $= s$, the normal BM $= n$, the ſub-normal PB $= q$, arch of any curve $= z$.

Then 1. $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = s$, or PT, the ſub-tangent, by means of which general expreſſion, and the equation expreſſing the relation between the abſciſſa and ordinate, the ratio of their Fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the length of the ſub-tangent PT.

Therefore by help of the equation of the curve Fig. 22. exterminate $\dot{x}$ or $\dot{y}$ out of the quantity $\frac{y\dot{x}}{\dot{y}}$, and you will get the value of PT the ſub-tangent.

2. $\dot{x} : \dot{y} :: y : \frac{y\dot{y}}{\dot{x}}$ = PB the ſub-normal $= q$.
3. $\dot{z} : \dot{y} :: t : y$ the ordinate.
4. $\dot{z} : \dot{x} :: t : s$ ſub-tangent.

And $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, therefore the Fluxion of the curve line AM is $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$. Then,

5. $\dot{y} : \sqrt{\dot{x}^2+\dot{y}^2} :: y : t$; and $t = \frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{y}}$.
6. $\dot{x} : \sqrt{\dot{x}^2+\dot{y}^2} :: y : n$; and $n = \frac{y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}$.
7. $\dot{x} : \dot{y} :: y : q$; and $q = \frac{y\dot{y}}{\dot{x}}$.

And $x + \frac{y\dot{y}}{\dot{x}} = \frac{y\dot{y}+x\dot{x}}{\dot{x}}$ = ſum abſciſſa and ſub-normal.

And becauſe $x + a = \frac{y\dot{x}}{\dot{y}}$, $a = \frac{\dot{y}x - x\dot{y}}{\dot{y}}$, where a = ſubtangent *leſs* abſciſſa.

The values of any of theſe being made, from the given conditions of a *Problem* propoſed, reſpectively equal to their values here, the nature of the curve will be found, and the contrary. Thus,

If $s = ny^n = \frac{y\dot{x}}{\dot{y}}$, then $n\dot{y}y^{n-1} = \dot{x}$, therefore $y^n = x$.

And if $y^n = x$, then $s = \frac{y\dot{x}}{\dot{y}} = ny^n = nx$.

Or if $px = yy$, then $p\dot{x} = 2y\dot{y}$, therefore $q = \frac{y\dot{y}}{\dot{x}} = \frac{yp\dot{x}}{2y\dot{x}} = \frac{p}{2} = \frac{1}{2}p$.

EXAM.

Fig.

EXAM. 1.

To draw a tangent to the circle.

23. Let $AP = x$, $PM = y$, radius $AB = a$, then $PC = 2a - x$, and by the property of the circle $AP \times PC = PM^2$, that is, $x \times \overline{2a - x}$, or $2ax - xx = yy$, which put into Fluxions is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$, divide by 2, it is $a\dot{x} - x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{a-x}$, but PT or $s = \frac{y\dot{x}}{\dot{y}} = \dot{x} \times \frac{y}{\dot{y}}$, then $\frac{y\dot{y}}{a-x} \times \frac{y}{\dot{y}} = \frac{y^2\dot{y}}{a\dot{y} - x\dot{y}} = \frac{y^2}{a-x}$, but $y^2 = 2ax - xx$, then $\frac{y^2}{a-x} = \frac{2ax - xx}{a-x} = PT$. Whence $a - x : y :: y : \frac{yy}{a-x} = \frac{2ax - xx}{a-x}$, as before.

EXAM. 2.

To draw a tangent to the ellipſis.

24. Let tranſverſe diameter $AB = 2a$, $p =$ parameter $= 1$, $AP = x$, $PM = y$, then $PB = 2a - x$; by property of the ellipſis $PB \times AP : PM^2 :: AB : p$, that is, as $\overline{2a - x} \times x : yy :: 2a : 1$. Then $2ax - xx = 2ayy$, which put into Fluxions is $2a\dot{x} - 2x\dot{x} = 4ay\dot{y}$, dividing by 2 is $a\dot{x} - x\dot{x} = 2ay\dot{y}$, and $\dot{x} = \frac{2ay\dot{y}}{a-x}$, but the ſub-tangent PT or $s = \frac{y\dot{x}}{\dot{y}} = \dot{x} \times \frac{y}{\dot{y}}$, then $\frac{2ay\dot{y}}{a-x} \times \frac{y}{\dot{y}} = \frac{2ay^2\dot{y}}{a\dot{y} - x\dot{y}} = \frac{2ay^2}{a-x}$, but $2ay^2 = 2ax - xx$, then $\frac{2ay^2}{a-x} = \frac{2ax - xx}{a-x} = PT$ the ſubtangent.

EXAM. 3.

To draw a tangent to the parabola.

Let $AP = x$, $PM = y$, parameter $= p = 1$, then $px = yy$, or $x = yy$ by nature of the parabola, Fig. which put into Fluxions is $\dot{x} = 2y\dot{y}$, but PT or s 22. $= \frac{y\dot{x}}{\dot{y}} = \dot{x} \times \frac{y}{\dot{y}}$, then $2y\dot{y} \times \frac{y}{\dot{y}} = \frac{2y^2\dot{y}}{\dot{y}} = 2yy$, but $x = yy$, then $2x = 2yy$, and therefore in the common parabola $2x =$ PT the ſubtangent = twice AP, its correſponding abſciſſa PM.

PROB. III.

To find the length of curve lines.

1. The ordinates in all curves reſpect either the axis, or a determined point, and where the ordinates are made parallel to one another, and right to the axis,

Let the curve AM $= z$, its Fluxion $= \dot{z}$. 22.
Abſciſſa AP $= x$, its Fluxion $= \dot{x}$.
Ordinate PM $= y$, its Fluxion $= \dot{y}$.
Then $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, as before.

Now when you have an equation expreſſing the ordinate to the abſciſſa, you may find the length of the curve thus.

Subſtitute in the general equation the value of the ſquare of the Fluxion of the abſciſſa or ordinate, deduced from the equation of the curve, in the room of the ſquare of the Fluxion of the abſciſſa or ordinate, and there will come out a new equation for the Fluxion of the curve, whoſe Fluent will be the length of the curve ſought.

2. Where the ordinates reſpect a point; let $r =$ radius of a circle deſcribed on that point, $x =$ abſciſſa, and $y =$ that part intercepted between the point and curve, then $\frac{\sqrt{y^2\dot{x}^2+r^2\dot{y}^2}}{r}$ will be the Fluxion of the curve. And by ſubſtituting in the room

Fig. 22. of $\dot{x}\dot{x}$ or $\dot{y}\dot{y}$ their values from the equation of the curve, there will be produced an equation, whoſe Fluent is the length required; that is,

By the nature of the curve exterminate $\dot{x}$ or $\dot{y}$ out of the equation $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, and the Fluent will give z, the length of the curve.

E X A M. 1.

Let $y = \frac{2}{3cc} \times \overline{cc+xx}|^{\frac{3}{2}}$ *be an equation to a curve, to find the length of that curve.*

1. $y = \frac{2}{3cc} \times \overline{cc+xx}|^{\frac{3}{2}}$, which put into Fluxions is $\dot{y} = \frac{2x\dot{x}}{cc}\sqrt{cc+xx}$, ſquared $\dot{y}\dot{y} = \frac{4x^2\dot{x}}{c^4} \times \overline{cc+xx}$ $= \frac{4c^2x^2\dot{x}+4x^4\dot{x}^2}{c^4} = \frac{4x^2\dot{x}^2}{cc} + \frac{4x^4\dot{x}^2}{c^4}$, but $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, the Fluxion of the curve, then for $\dot{y}\dot{y}$, ſubſtitute its equal, and $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} = \dot{x}\sqrt{1+\frac{4xx}{cc}+\frac{4x^4}{c^4}} = \dot{x} + \frac{2x^2\dot{x}}{cc}$, whoſe Fluent is $z = x + \frac{2x^3}{3cc}$ the length of the curve.

E X A M. 2.

To find the length of the arch AM *of the common parabola.*

25. Put AP $= x$, PM $= y$, parameter $= a$, then $ax = yy$ by property of the parabola, which put into Fluxions is $a\dot{x} = 2y\dot{y}$, whence $\dot{x} = \frac{2y\dot{y}}{a}$, which ſquared is $\dot{x}\dot{x} = \frac{4y^2\dot{y}^2}{aa}$, now the Fluxion of the curve is $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, for $\dot{x}\dot{x}$ ſubſtitute $\frac{4y^2\dot{y}^2}{aa}$ its

equal in the equation, then $\dot{z} = (\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} =)$ Fig. 25. $\sqrt{\frac{4y^2\dot{y}^2}{aa} + \dot{y}\dot{y}} = \frac{\sqrt{4y^2\dot{y}^2 + aa\dot{y}\dot{y}}}{a} = \frac{\dot{y}\sqrt{4y^2+aa}}{a} =$ $\frac{\dot{y}}{a}\sqrt{4yy+aa}$ for the Fluxion of the curve.

Now $\sqrt{aa+4yy} = a + \frac{4y^2}{2a} - \frac{16y^4}{8a^3} + \frac{64y^6}{16a^5} -$ &c. $= a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5}$ &c. which multiplied by $\frac{\dot{y}}{a}$ is $\dot{z} = \dot{y} + \frac{2y^2\dot{y}}{aa} - \frac{2y^4\dot{y}}{a^4} + \frac{4y^6\dot{y}}{a^6}$ &c. whoſe Fluent is $z = y + \frac{2y^3}{3aa} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6}$ &c.

Or,

The Fluent of $\frac{\dot{y}}{a}\sqrt{aa+4yy}$ is $= \frac{y}{a}\sqrt{\frac{aa}{4}+yy}$ $+ \frac{2.30258a}{4} \times \log. y + \frac{\sqrt{\frac{aa}{4}+yy}}{\frac{1}{2}a}.$

Or thus.

Since $ax = yy$, therefore $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} = \frac{\dot{y}}{a}$ $\sqrt{a^2+4y^2}$, whoſe Fluent is $\frac{Ay}{2a} + .575646273a \times$ $\log. \frac{2y+A}{a}$, the length of the ſemi-curve, where $A = \sqrt{a^2+4y^2}$.

In Numbers.

Given the abſciſſa $= 4$ feet, and ordinate rightly applied $= 6$ feet, required the curve's length.

When abſciſſa $= 4$, ſemi-ordinate $= 3$, that is $y = 3$, $a = 2\frac{1}{4}$, then $\frac{Ay}{2a} + .575646273a \times \log.$

$2y+A$

Fig. 25. $\frac{2y+A}{a} = 5.232421$, and the whole curve $=$ 10.464842 feet.

EXAM. 3.

Let $ax^2 = y^3$, which is an equation for a cubic parabola. In Fluxions $2ax\dot{x} = 3y^2\dot{y}$, whence $\dot{x} = \frac{3y^2\dot{y}}{2ax}$, ſquared is $\dot{x}\dot{x} = \frac{9y^4\dot{y}^2}{4a^2x^2}$, but $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, in which for $\dot{x}\dot{x}$ ſubſtitute $\frac{9y^4\dot{y}^2}{4a^2x^2}$ its equal, then $\dot{z}\dot{z} = \frac{9y^4\dot{y}^2}{4a^2x^2} + \dot{y}\dot{y}$, alſo $ax^2 = y^3$, which multiplied by 4, is $4ax^2 = 4y^3$, then by ſubſtituting $4y^3$ for $4ax^2$, $\dot{z}\dot{z} = \left(\frac{9y^4\dot{y}^2}{4a^2x^2} + \dot{y}\dot{y}\right) = \frac{9y^4\dot{y}^2}{4ay^3} + \dot{y}\dot{y} = \frac{9y\dot{y}^2}{4a} + \dot{y}\dot{y} = \frac{9y\dot{y}^2 + 4a\dot{y}\dot{y}}{4a} = \frac{4a+9y}{4a}\dot{y}\dot{y} = \overline{1 + \frac{9y}{4a}} \times \dot{y}\dot{y}$, then $\dot{z} = \dot{y}\sqrt{1 + \frac{9y}{4a}}$, the Fluxion of the arch, whoſe Fluent is (by ſuppoſing $a = 1$) $= \frac{1}{27} \times \overline{4+9y}\sqrt{4+9y}$.

Or thus.

But $\dot{z} = \dot{y}\sqrt{1 + \frac{9y}{4a}} = \overline{1 + \frac{9y}{4a}}\Big|^{\frac{1}{2}}\dot{y}$. Comparing this with Caſe 9, Form 3, we have $\alpha = 1$, $\beta = \frac{9}{4a}$, $z = y$, $\mu = \frac{1}{2}$,

Then $\frac{\overline{\alpha + \beta z}\big|^{\mu+1}}{\mu+1 . \beta} = \frac{\overline{1+\frac{9y}{4a}}\big|^{\frac{3}{2}}}{\frac{3}{2} \times \frac{9}{4a}} = \frac{\overline{1+\frac{9y}{4a}}\big|^{\frac{3}{2}}}{\frac{27}{8a}} = \frac{8a}{27} \times \overline{1 + \frac{9y}{4a}}\Big|^{\frac{3}{2}}$.

Now to find whether any thing is to be added or taken from it, it muſt be obſerved, that in the vertex $y = 0$, then $\frac{4}{27}\sqrt{4} = \frac{4}{27} \times 2 = \frac{8}{27}$ to be taken from it, and by reſtoring a, the correct Fluent is Fig. 26.

$$z = \frac{8a}{27} \times \overline{1 + \frac{9y}{4a}}\Big|^{\frac{3}{2}} - \frac{8a}{27}.$$

EXAM. 4.

To find the length of the arch of the circle AM.

Let $AP = x$, $PM = y =$ ſine of 30° $= .5 =$ half the ſide of the hexagon, radius $= 1$. By property of the circle $AP \times BP = \overline{PM}^2$, that is $2x - xx = yy$, in Fluxions $2\dot{x} - 2x\dot{x} = 2y\dot{y}$, dividing by 2, it is $\dot{x} - x\dot{x} = y\dot{y}$, hence $\dot{x} = \frac{y\dot{y}}{1-x}$, ſquared is $\dot{x}\dot{x} = \frac{y^2\dot{y}^2}{xx-2x+1} = \frac{y^2\dot{y}^2}{1-yy}$ (for $2x - xx = yy$). The Fluxion of the arch AM is $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} = \sqrt{\frac{y^2\dot{y}^2}{1-yy} + \dot{y}^2} = \sqrt{\frac{y^2\dot{y}^2+\dot{y}^2-y^2\dot{y}^2}{1-yy}}$ $= \sqrt{\frac{\dot{y}^2}{1-yy}} = \dot{y}\sqrt{\frac{1}{1-yy}}$.

Now $\frac{1}{1-yy} = 1 + y^2 + y^4 + y^6 + y^8$ &c. by diviſion, whoſe ſquare root is $= 1 + \frac{y^2}{2} + \frac{3y^4}{8} + \frac{5y^6}{16} + \frac{35y^8}{128}$ &c. which multiplied by $\dot{y}$, produces $\dot{y} + \frac{y^2\dot{y}}{2} + \frac{3y^4\dot{y}}{8} + \frac{5y^6\dot{y}}{16} + \frac{35y^8\dot{y}}{128}$ &c. whoſe Fluent is $z = y + \frac{y^3}{2.3} + \frac{3y^5}{5.8} + \frac{5y^7}{7.16} + \frac{35y^9}{9.128}$ &c. the length of the arch AM.

$y =$

Fig. 26.

$y = .5$

$\frac{y^3}{2.3} = .020833$ &c.

$\frac{3y^5}{5.8} = .002344$ &c.

$\frac{5y^7}{7.16} = .000348$ &c.

$\frac{35y^9}{9.128} = .000059$ &c.

$= .523585$ &c. a ſixth part of the circumference, rad. $= 1$.

$\times\ 6$

3.1415 circumference, diam. $= 1$.

Or thus.

Let AC the radius $= r$, AT $= t$, T$t = \dot{t}$, AM $= z$, M$o = \dot{z}$, then CT $= \sqrt{rr+tt}$; by the ſimilar triangles CAT, cTt, we have CT : Ac :: tT : Tc, that is $\sqrt{r^2+t^2} : r :: \dot{t} : \frac{r\dot{t}}{\sqrt{rr+tt}} =$ Tc; and CT : Tc :: CM : Mo, that is, $\sqrt{rr+tt} : \frac{r\dot{t}}{\sqrt{rr+tt}} :: r : \frac{r^2\dot{t}}{r^2+t^2} = \dot{z}$, the Fluxion of the arch in the terms of the tangent.

Now $\frac{r^2}{r^2+t^2} = 1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8}$ &c. by diviſion, which multiply by $\dot{t}$ produces $\dot{t} - \frac{t^2\dot{t}}{r^2} + \frac{t^4\dot{t}}{r^4} - \frac{t^6\dot{t}}{r^6} + \frac{t^8\dot{t}}{r^8}$ &c. $= \dot{z}$, the Fluent $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}$ &c. length of the arch AM.

Let radius AC = CM = 1, then $z = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9}$ &c. and let arch AM = 30 degrees, then its tangent is $= \sqrt{\frac{1}{3}}$, for the ſine of 30 degrees is $\frac{1}{2}$, = half the chord of 60°, which is = radius, therefore the coſine is $= \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}$. And as coſine : ſine : : radius : tangent, that is, $\sqrt{\frac{3}{4}} : \frac{1}{2} : : 1 : \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \sqrt{\frac{1}{3}} = .577350269$ &c. $= t$. Then $t^2 = \frac{1}{3}$, $t^3 = \frac{1}{3}\sqrt{\frac{1}{3}}$, $t^5 = \frac{1}{9}\sqrt{\frac{1}{3}}$, $t^7 = \frac{1}{27}\sqrt{\frac{1}{3}}$, $t^9 = \frac{1}{81}\sqrt{\frac{1}{3}}$ &c. which values being ſubſtituted for t, we get AM, or $z = \sqrt{\frac{1}{3}} - \frac{1}{9}\sqrt{\frac{1}{3}} + \frac{1}{45}\sqrt{\frac{1}{3}} - \frac{1}{189}\sqrt{\frac{1}{3}} + \frac{1}{729}\sqrt{\frac{1}{3}} - \frac{1}{2673}\sqrt{\frac{1}{3}} +$ &c. $= \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 9\sqrt{3}} - \frac{1}{7 \cdot 27\sqrt{3}} + \frac{1}{9 \cdot 81\sqrt{3}} - \frac{1}{11 \cdot 243\sqrt{3}}$ &c. $= \frac{1}{12}$ of the circumference. Now ſix times this muſt be the length of the ſemi-circumference; then put $6\sqrt{\frac{1}{3}} = \sqrt{12} = \text{A}$, $\frac{\text{A}}{3 \cdot 3} = \text{B}$, $\frac{3\text{B}}{5 \cdot 3} = \text{C}$, $\frac{5\text{C}}{7 \cdot 3} = \text{D}$, $\frac{7\text{D}}{9 \cdot 3} = \text{E}$, &c. Then half the circumference is $= \text{A} - \text{B} + \text{C} - \text{D} + \text{E} - \text{F} + \text{G}$ &c. Fig. 26.

Fig. 26.

$$\begin{array}{rlrrlr}
\sqrt{12} = & A = & 3.4641.0161.5 & - & B = & .3849.0017.9 \\
 & C = & 769.8003.6 & - & D = & 183.2858.0 \\
 & E = & 47.5185.4 & - & F = & 12.9596.0 \\
 & G = & 3.6552.7 & - & H = & 1.0559.7 \\
 & I = & 3105.8 & - & K = & 926.3 \\
 & L = & 279.4 & - & M = & 85.0 \\
 & N = & 26.1 & - & O = & 8.0 \\
 & P = & 2.5 & - & Q = & 8 \\
\cline{6-6}
 & R = & 2 & - & & .4046.4051.7 \\
 & \&c. & & & & \\
\cline{3-3}
 & + & 3.5462.3317.2 & & & \\
 & - & .4046.4051.7 & & & \\
\cline{2-3}
\end{array}$$

3.1415.9265.5 = half the circumference, radius = 1, or the circumference, diameter = 1.

There are other methods for finding the circumference of a circle, particularly one communicated to me by the ingenious Mr. *William Brown*, of *Cleobury*, *Shropſhire*. If the diameter of a circle be 1, the circumference will be $\frac{72623\sqrt{3}}{40040} +$

$$\underset{A}{\frac{\sqrt{3}}{11.12.13.15}} - \underset{B}{\frac{1A}{17.3}} + \underset{C}{\frac{3B}{19.3}} - \underset{D}{\frac{5C}{21.3}} + \underset{E}{\frac{7D}{23\ 3}} - \underset{F}{\frac{9E}{25.3}}$$

&c. Here the firſt term will give ſix figures true, and if we make uſe of nine terms, we ſhall get 14 figures true, &c.

I will here ſhew how the circumference may be found from my edition of Mr. *Stirling*'s *Differential Method*.

To find the ſum of the ſeries $z - \frac{z^3}{2} + \frac{z^5}{5} - \frac{z^7}{7}$ *&c. when* $z = 1$, *which expreſſes the length of the arch of* 45 *degrees to the radius* 1.

The given ſeries is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c. Fig. 26.
or $1 - \frac{1}{3}A - \frac{3}{5}B - \frac{5}{7}C - \frac{7}{9}D$, &c.

Compare this with *Prop.* VII. *page* 35, and the ſucceſſive values of z will be $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. and $n = 1$, $m - 1 = 1$, or $m = 2$, whence $S = \frac{1}{2}T + \frac{A}{2z} + \frac{2B}{2z+2} + \frac{3C}{2z+4} + \frac{4D}{2z+6}$ &c. ſum up the ſix firſt terms of the given ſeries, which is $= .744012$, and put $T = \frac{1}{13}$ the ſeventh term, and then $z = 7\frac{1}{2}$, then we have $S = \frac{1}{26} + \frac{A}{15} + \frac{2B}{17} + \frac{3C}{19} + \frac{4D}{21}$ &c. that is,

$$\begin{array}{llr}
 & S = & .038462 \\
 & & 2564 \\
 & & 301 \\
 & & 48 \\
 & & 9 \\
 & & 2 \\
\hline
 & & .041386 \\
\text{To which add} & & .744012 \\
\hline
\text{The ſum} & & .785398
\end{array}$$

.785398 is $\frac{1}{8}$ the periphery of the circle when radius $= 1$.

$$\begin{array}{lr}
\text{Then mult.} & 8 \\
\hline
 & 2)6.283184 \\
\hline
 & 3.141592
\end{array}$$

2)6.283184 circumf. when rad. is 1.

3.141592 circumference when diameter is 1. Which by taking only ſix terms of the laſt ſeries is more exact than if *one hundred thouſand* terms of the given ſeries had been uſed.

The Reader would do well to conſult *Ex.* 1, p. 40, *Ex.* 2, p. 49, of my edition of Mr. *Stirling's Differential Method.*

PROB. IV.

Given the radius and the arch of a circle to find its ſine.

27. Put radius $CA = r$, ſine $DB = x$, then will the coſine $CB = \sqrt{rr - xx}$, put the given arch $AD = z$, draw C*d* infinitely near to CD, *bd* infinitely near and parallel to BD, and *m*D parallel to *b*B, then is *m*D or $\dot{x}$ the Fluxion of the ſine; and D*d* or $\dot{z}$ the Fluxion of the arch, and by the ſimilar triangles DCB, *dm*D, it will be CB : CD : : D*m* : D*d*; that is, $\sqrt{rr - xx} : r : : \dot{x} : \dot{z}$, whence $\dot{z} = \frac{r\dot{x}}{\sqrt{rr - xx}}$, which ſquared is $\dot{z}\dot{z} = \frac{r^2\dot{x}^2}{rr - xx}$, and $r^2 - x^2 = \frac{r^2\dot{x}\dot{x}}{\dot{z}\dot{z}}$. Now put $x = Az + Bz^3 + Cz^5$ &c. which put into Fluxions is $\dot{x} = A\dot{z} + 3Bz^2\dot{z} + 5Cz^4\dot{z}$ &c. ſquared is

$$\dot{x}\dot{x} = A^2\dot{z}^2 + 6ABz^2\dot{z}^2 \left.\begin{matrix} + 9BBz^4\dot{z}^2 \\ + 10ACz^4\dot{z}^2 \end{matrix}\right\} \text{\&c.}$$

Alſo

$$xx = AAz^2 + 2ABz^4 \left.\begin{matrix} + BBz^6 \\ + 2ACz^6 \end{matrix}\right\} \text{\&c.}$$

Whence $rr - xx = rr - A^2z^2 - 2ABz^4$ &c. and

$$\frac{r^2\dot{x}^2}{\dot{z}\dot{z}} = rrAA + 6rrABz^2 \left.\begin{matrix} + 9rrBBz^4 \\ + 10ACz^4 \end{matrix}\right\} \text{\&c.}$$

wherefore theſe two ſeries are equal, and the value of A, B, C, &c. is found by equating the homologous powers of theſe ſeries, therefore it will be

$$-rr - AAzz - 2ABz^4 \text{ \&c.} = rrAA + 6rrABz^2 \left.\begin{matrix} + 9r^2B^2z^4 \\ + 10r^2ACz^4 \end{matrix}\right\} \text{\&c.}$$

Whence $rr = rrAA$, and therefore $A = 1$; alſo $-AAz^2 = 6rrABz^2$, whence $-1 = 6rrB$, and $B = \frac{-1}{6rr}$; alſo $-2ABz^4 =$

$9r^2B^2z^4+10r^2ACz^4$, or by ſubſtitution, and dividing by z^4, it is $+\frac{2}{6rr}=\frac{9}{36rr}+10r^2AC$, or $10r^2AC=\frac{3}{36rr}$, and $C=\frac{3}{10.36r^4}=\frac{1}{2.3.4.5r^4}$ &c. whence the ſeries $Az+Bz^3+Cz^5$ &c. $=x$, becomes $x=z-\frac{z^3}{2.3r^2}+\frac{z^5}{2.3.4.5r^4}-\frac{rz^7}{2.3.4.5.6.7r^6}$ &c. whoſe law of continuation is manifeſt, and converges faſt, the arch being ſmall. Fig. 27.

PROB. V.

Given the radius and the arch of a circle to find the ſecant.

Let as before $CA=r$, ſecant $CB=y$, then the tangent $AB=\sqrt{yy-rr}$, put the given arch $AD=z$, its Fluxion $Dd=\dot z$; draw Ca infinitely near to CB, from C with the radius CB, deſcribe the arch Bb, then ba will be the Fluxion of the required ſecant, or $\dot y$, whence the arcs Dd, Bd, will be as their radii CD, CB; then $r:y::\dot z:\frac{y\dot z}{r}=Bb$, and by the ſimilar triangles BCA, Bab; CA : AB :: Bb : ba, that is, $r:\sqrt{yy-rr}::\frac{y\dot z}{r}:\dot y$, and ſquaring, $rr:yy-rr::\frac{y^2\dot z\dot z}{rr}:\dot y\dot y$; whence $r^2\dot y^2=\frac{y^4\dot z^2-r^2y^2\dot z^2}{r^2}$, and $\frac{r^4\dot y^2}{\dot z\dot z}=y^4-r^2y^2$. Put $y=A+Bz^2+Cz^4+Dz^6$ &c. which in Fluxions is $\dot y=2Bz\dot z+4Cz^3\dot z+6Dz^5\dot z$ &c. which ſquared we have $\dot y\dot y=4B^2z^2\dot z^2+16BCz^4\dot z^2+\left.\begin{array}{l}16C^2z^6\dot z^2\\+24DBz^6\dot z^2\end{array}\right\}$ &c. 28.

Fig. 28. The firſt ſquared is $y^2 = A^2 + 2ABz^2 + \left.\begin{matrix} 2ACz^4 \\ +BBz^4 \end{matrix}\right\}$ &c.

Squared again is $y^4 = A^4 + 4A^3Bz^2 + \left.\begin{matrix} 6A^2B^2z^4 \\ +4A^3Cz^4 \end{matrix}\right\}$ &c.

Therefore $\frac{r^4\dot{y}^2}{\dot{z}\dot{z}} = 4r^4B^2z^2 + 16r^4BCz^4$ $\left.\begin{matrix} +16r^4C^2z^6 \\ +24r^4DBz^6 \end{matrix}\right\}$ &c.

And $y^4 - r^2y^2 = \left.\begin{matrix} A^4 & +4A^3Bz^2 & +6A^2B^2z^4 \\ -r^2A^2 & -2r^2ABz^2 & +4A^3Cz^4 \\ & & -2r^2ACz^4 \\ & & -r^2BBz^4 \end{matrix}\right\}$ &c.

Whence equating the homologous powers we get $A^4 - r^2A^2 = 0$, therefore $A^2 = r^2$, and $A = r$.

Alſo $4r^4B^2z^2 = 4A^3Bz^2 - 2r^2ABz^2$, dividing by Bz^2, and putting r in the room of A, we have $4r^4B = 4r^3 - 2r^3$, therefore $B = \frac{1}{2r}$, alſo $16r^4BCz^4 = 6A^4B^2z^4 + 4A^3Cz^4 - 2r^2ACz^4 - r^2B^2z^4$, which divided by z^4, and ſubſtituting the values of A and B, we have $8r^3C = \frac{6}{4} + 4r^3C - 2r^3C - \frac{1}{4}$, whence $6r^3C = \frac{5}{4}$, and $C = \frac{5}{2\ 3.4r^3}$ &c. Therefore the ſeries $A + Bz^2 + Cz^4$ &c. expreſſing the value of the ſecant becomes $y = r + \frac{zz}{2r} + \frac{5z^4}{2\ 3\ 4r^3} + \frac{61z^6}{720r^5}$ &c. which converges faſt when the arch is ſmall, as in the preſent caſe.

Of the circular arch, its ſines, tangent and chord.

26. Let C be the center, AC the radius $= a$, the arch MsA $= v$, ſine MP $= z$, coſine PC $= x$, verſed ſine PA $= y$, tangent AT $= t$, ſecant CT $= s$, and chord MA $= c$, it is propoſed to ſhew how to find the arch from the given ſine, tangent or chord, and *vice verſâ*.

LEMMA

LEMMA. I.

To find the relation between $\dot{v}$ and $\dot{y}$.

Since $\dot{v}^2 = \dot{y}^2 + \dot{z}^2$, therefore $2ay - yy = zz$ by nature of the circle, by the help of which equation exterminate $\dot{z}$, then $\dot{v}^2 = \frac{aa\dot{y}\dot{y}}{2ay-yy}$, whence $\dot{v} = \frac{a\dot{y}}{\sqrt{2ay-yy}}$ or $a\dot{y} \times \overline{2ay-yy}\big|^{-\frac{1}{2}}$. Q. E. I.

COR. I.

Alſo $\dot{v} = a\dot{z} \times \overline{aa-zz}\big|^{-\frac{1}{2}}$ or $\frac{a\dot{z}}{\sqrt{aa-zz}}$, as will be evident by exterminating y and $\dot{y}$ out of the equation of the above *Lemma*.

COR. 2.

Again $\dot{v} = \frac{a^2\dot{t}}{a^2+t^2}$ or $a^2\dot{t} \times \overline{aa+tt}\big|^{-1}$. For CP : PM : : CA : AT; that is, $a - y : \sqrt{2ay-yy} : : a : t$, whence $t = \frac{a\sqrt{2ay-yy}}{a-y}$, by help of which y and $\dot{y}$ from the *Lemma* may be exterminated, and the Corollary will be evident.

COR. 3.

Again, $\dot{v} = \frac{2a\dot{c}}{\sqrt{4aa-cc}}$ or $2a\dot{c} \times \overline{4a^2-c^2}\big|^{-\frac{1}{2}}$, as will be evident, if from the *Lemma* y and $\dot{y}$ be exterminated by the help of this equation $2ay = cc$.

PROB. VI.

Given (a) the radius and (y) the verſed ſine to find (v) the arch thereof.

Since by the preceding *Lemma* $\dot{v} = a\dot{y} \times \overline{2ay-yy}\big|^{-\frac{1}{2}}$; therefore $\frac{1}{\sqrt{2ay-yy}} = \frac{1}{\sqrt{2ay}} \times : 1$

$+ \frac{y}{4a} + \frac{3y^2}{4.8a^2} + \frac{3.5y^3}{4.8.12a^3} + \frac{3.5.7y^4}{4.8.12.16a^4}$ &c. which multiplied by $a\dot{y}$, gives the Fluxion of the arch in the terms of the verſed ſine, whoſe Fluent is $v = \sqrt{2ay} \times : 1 + \frac{y}{2.3.2a} + \frac{3y^2}{2.4.5.4a^2} + \frac{3.5y^3}{2.4.6.7.8a^3}$ &c.

PROB. VII.

Given the radius and the ſine to find the arch.

Since $\dot{v} = \frac{a\dot{z}}{\sqrt{aa-zz}}$ or $a\dot{z} \times \overline{aa-zz}|^{-\frac{1}{2}}$, therefore $\frac{1}{\sqrt{aa-zz}}$ put into a ſeries, and then multiplied by $a\dot{z}$, the Fluent is $v = z + \frac{z^3}{2.3aa} + \frac{3.z^5}{2.4.5a^4} + \frac{3.5z^7}{2.4.6.7a^6}$ &c.

PROB. VIII.

Given the radius and tangent of an arch to find the arch.

By *Cor.* 2, $\dot{v} = \frac{a^2\dot{t}}{aa+tt}$ or $a^2\dot{t} \times \overline{aa+tt}|^{-1} = a^2\dot{t}$ $\frac{1}{aa} - \frac{tt}{a^4} + \frac{t^4}{a^6} - \frac{t^6}{a^8}$ &c. $= \dot{t} - \frac{t^2\dot{t}}{aa} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6} + \frac{t^8\dot{t}}{a^8}$ &c. Fluent is $v = t - \frac{t^3}{3aa} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ &c.

PROB.

PROB. IX.

Given the radius and the chord to find the arch.

Since by *Cor.* 3, $\dot{v} = \frac{2a\dot{c}}{\sqrt{4a^2-cc}}$ or $2a\dot{c} \times \overline{4a^2-cc}|^{-\frac{1}{2}}$, now $\frac{1}{\sqrt{4a^2-cc}}$ put into a ſeries, and multiplied by $2a\dot{c}$, gives the Fluxion of the arch, whoſe Fluent is $v = c + \frac{c^3}{\overline{2}|^3.3.aa} + \frac{3c^5}{2^5..4.5.a^4} + \frac{35.c^7}{2^7.4.6.7.a^6}$ &c.

SCHOLIUM.

In the ſame manner v the arch may be found from x or s, the given coſine or ſecant.

COROLLARY.

If v the arch be very ſmall, then the ſine, tangent, chord and arch, are very nearly equal. For in this caſe all the terms of the ſeries except the firſt are of ſmall value, and may be neglected, and therefore $v = z$ by *Prob.* VII. $v = t$ by *Prob.* VIII. and $v = c$ by *Prob.* IX.

PROB. X.

Given the radius and the arch to find the tangent.

Since $\dot{v} \times \overline{aa+tt} = aa\dot{t}$ by *Cor.* 2, therefore aſſume $t = Av + Bv^2 + Cv^3 + Dv^4 + Ev^5$ &c. whence by comparing the homologous terms we find

$A = 1$, $B = 0$, $C = \frac{1}{3aa}$, $D = 0$, $E = \frac{2}{3.5a^4}$ &c.

whence $t = v + \frac{v^3}{3aa} + \frac{2v^5}{3.5a^4}$ &c. Q. E. I.

SCHOL.

SCHOLIUM.

Likewiſe from the radius and the arch being given, you may find the ſine, coſine, verſed ſine, tangent, and chord of that arch.

Thus to find z the ſine. Since $\dot{v}\sqrt{aa-zz}=a\dot{z}$, then aſſume $z = Av + Bv^2 + Cv^3 + Dv^4 + Ev^5$ &c. and by comparing the homologous terms, I find $A = 1$, $B = 0$, $C = \frac{-1}{2.3a^2}$, $D = 0$, $E = \frac{1}{2.3.4.5a^4}$, whence $z = v - \frac{v^3}{2.3a^2} + \frac{v^5}{2.3.4.5a^4} - \frac{v^7}{2.3.4.5.6.7a^6}$ &c.

COROLLARY,

Hence is deduced a genuine and eaſy method for computing tables of ſines, tangents, ſecants, &c.

1. Let radius $a = 1.00000$ &c. then the circumference of the whole circle will be 6.2831853 &c. $= p$.

2. To find the ratio between the radius and any arch denoted by degrees or minutes. As the whole circumference in minutes, that is, as 21600′ to any arch denoted by minutes, ſo is 6.2831853, to an arch denoted by parts of the radius. *Exam.* To find the ratio between the radius and arch of one minute; the proportion will be as 21600′ : 1′ : : 6.2831853 : .0002908, therefore the radius is to the arch of 1 min. as 1.00000 to .0002908.

3. To find the right ſine of an arch of one minute; ſince $a = 1$, and $v = .0002908$, therefore $z = .0002908 - \frac{\overline{.0002908}|^3}{6} + \frac{\overline{.0002908}|^5}{120}$ &c. by the ſeries in the foregoing Scholium. You will find the right ſine of any arch in the ſame manner. And thus a table of ſines will be computed. And if in the ſeries of the laſt *Prob.* .0002908 be ſub-

ſtituted for v, then the tangent of the arch of one minute will be $t = .0002908 + \frac{\overline{.0002908}|^3}{3}$ &c. and thus a table of tangents will be computed; but tangents, ſecants, &c. will be eaſier found from the computation of right ſines, as is well known. Fig.

EXAM. 5.

To find the length of the arch of an ellipſis.

Put AD $= t$, DC $= c$, AP $= x$, PM $= y$, $z =$ arch AM, then PB $= 2t - x$, and from the property of the ellipſis we have AP × PB : PM² : : AD² : DC², that is, $2tx - xx : yy :: tt : cc$, whence $\overline{2tx-xx} \times cc = ttyy$ and $2tx - xx = \frac{ttyy}{cc} = \frac{tt}{cc} \times yy$, which in Fluxions is $2t\dot{x} - 2x\dot{x} = \frac{tt}{cc} \times 2y\dot{y}$, dividing by 2, is $t\dot{x} - x\dot{x} = \frac{tt}{cc} y\dot{y} = \frac{tty\dot{y}}{cc}$, and $\dot{x} = \frac{tty\dot{y}}{cc \times \overline{t-x}} = \frac{tty\dot{y}}{tc^2 - c^2x}$, ſquared is $\dot{x}\dot{x} = \frac{t^4y^2\dot{y}^2}{t^2c^4 - 2tc^4x + c^4x^2} = \frac{t^4y^2\dot{y}^2}{c^4tt - c^2t^2y^2}$ (for $cc \times \overline{2tx-xx} = ttyy$, and $c^4 \times \overline{2tx-xx} = t^2c^2y^2$) now ſubſtitute $t^2c^2y^2$ for $\overline{2tx-xx} \times c^4$, then $\dot{x}\dot{x} = \frac{t^4y^2\dot{y}\dot{y}}{c^4tt - c^2t^2y^2}$, divide by tt, is $\dot{x}\dot{x} = \frac{t^2y^2\dot{y}\dot{y}}{c^4 - c^2y^2}$; but $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, then $\dot{z}\dot{z} = \frac{t^2y^2\dot{y}^2}{c^4 - c^2y^2} + \dot{y}\dot{y}$ (by ſubſtituting $\frac{t^2y^2\dot{y}\dot{y}}{c^4 - c^2y^2}$ for $\dot{x}\dot{x}$, in $\dot{x}\dot{x} + \dot{y}\dot{y}$) $= \frac{t^2y^2\dot{y}^2 + c^4\dot{y}^2 - c^2y^2\dot{y}^2}{c^4 - c^2y^2} =$ 29.

$c^4 +$

Fig. 29. $\frac{\overline{c^4+\overline{t^2-c^2}\times yy}}{c^4-c^2y^2}\,\dot{y}\dot{y} = \frac{\overline{c^4+\overline{tt-cc}\times yy}}{c^2-y^2} \times \frac{\dot{y}\dot{y}}{cc}$; therefore $\dot{z} = \frac{\dot{y}}{c}\sqrt{\frac{c^4+\overline{tt-cc}\times yy}{cc-yy}}$.

Let $c=1$, and put $d = tt-cc$, or $tt-1$, then $\dot{z} = \dot{y}\sqrt{\frac{1+dy^2}{1-yy}}$, the ſquare root of $1+dyy$ is $= 1+\frac{dy^2}{2}-\frac{d^2y^4}{8}+\frac{d^3y^6}{16}-\frac{5d^4y^8}{128}$ &c. And ſquare root of $1-yy$ is $= 1-\frac{y^2}{2}-\frac{y^4}{8}-\frac{y^6}{16}-\frac{5y^8}{128}$ &c. the former divided by the latter is $= 1+\frac{dy^2}{2}+\frac{dy^4}{2}-\frac{d^2y^4}{8}+\frac{dy^6}{2}-\frac{d^2y^6}{4}+\frac{d^3y^6}{16}$ &c. or $1+\frac{d}{2}y+\overline{\frac{d}{2}-\frac{dd}{8}}\,y^4+\overline{\frac{d}{2}-\frac{dd}{4}+\frac{d^3}{16}}\,y^6$ &c. which multiplied by $\dot{y}$, is $\dot{y}+\frac{d}{2}y\dot{y}+\overline{\frac{d}{2}-\frac{dd}{8}}\,y^4\dot{y}+\overline{\frac{d}{2}-\frac{dd}{4}+\frac{d^3}{16}}\,y^6\dot{y}$ &c. whoſe Fluent is $= y+\frac{d}{2.3}y^3+\overline{\frac{d}{2.5}-\frac{d^2}{5\;8}}\,y^5+\overline{\frac{d}{2.7}-\frac{d^2}{4.7}+\frac{d^3}{7.16}}\,y^7$ &c. for the arch AM. Now when AP flows to D, then y becomes $=$ C.

Or to find the arch MC.

Put AD $=t$, CD $=c$, PD $=x$, PM $=y$, then AP $=t-x$, and PB $=t+x$, then PB $\times$ AP : PM^2 : : AD^2 : DC^2, that is, $t^2-x^2 : yy :: t : c^2$, whence $c^2t^2-c^2x^2=t^2y^2$, and tranſpoſing is $c^2x^2=c^2t^2-t^2y^2$, in Fluxions $2c^2x\dot{x} = -2t^2y\dot{y}$, and $\dot{x} = \frac{tty\dot{y}}{c^2x}$

ſquared is $\dot{x}\dot{x} = \frac{t^4y^2\dot{y}^2}{c^4xx}$; but $t^2c^2 - c^2x^2 = t^2y^2$, therefore $xx = \frac{c^2t^2 - t^2y^2}{cc}$, and $c^4xx = c^4t^2 - c^2t^2yy$; now for c^4x^2, ſubſtitute its equal, then $\dot{x}\dot{x} = \frac{t^4y^2\dot{y}^2}{c^4t^2 - c^2t^2y^2}$ divide by tt, then $\dot{x}\dot{x} = \frac{t^2y^2\dot{y}^2}{c^4 - c^2y^2}$; but $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, then for $\dot{x}\dot{x}$ ſubſtitute its equal above, and $\dot{z}\dot{z} = \frac{t^2y^2\dot{y}^2}{c^4 - c^2y^2} + \dot{y}\dot{y}$, and conſequently $\dot{z} = \frac{\dot{y}}{c}\sqrt{\frac{c^4 + \overline{tt - cc} \times yy}{c^2 - y^2}}$, whoſe Fluent is (by putting $d = tt - cc$, or $d = tt - 1$) $z = y + \frac{d}{2.3}y^3 + \overline{\frac{d}{2.5} - \frac{dd}{5.8}}\,y^5 + \overline{\frac{d}{2.7} - \frac{d^2}{4.7} + \frac{d^3}{7.16}}\,y^7$ &c. the very ſame as above.

Perhaps it will not be improper to ſubjoin here another method for finding the periphery of an ellipſis, which was communicated to me by the ingenious Mr. *Brown* of *Cleobury*. This method of Mr. *Brown*'s, or rather the late Mr. *Thacker*'s for finding the arch of an ellipſe, is an inſtance of the great uſe of comparing curves or Fluents one with another, the doctrine whereof is delivered in the VIIth Prop. of *Sir Iſaac Newton*'s Quadratures; this is the inveſtigation of the firſt caſe thereof, applied to find the length of any arch of an ellipſis, by comparing it with the correſponding arch of the circumſcribed circle, from whence Mr. *Cunn* deduced his ſeries for the ſame purpoſe, and which method I have had by me for near 40 years.

EXAM. 6.

30. *To find the elliptic arch* DE *whoſe equation is* $\frac{c}{t}\sqrt{t^2-x^2}=y$; (where $t=$ CA, $c=$ CD, $x=$ CB $=$ FE, and $y=$ BE $=$ CF) which put into Fluxions gives $\frac{-cx\dot{x}}{t\sqrt{t^2-x^2}}=\dot{y}$, therefore $\frac{c^2x^2\dot{x}\dot{x}}{t^4-t^2x^2}=\dot{y}^2$, but $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}\sqrt{t^2-dx^2}}{\sqrt{t^2-x^2}}=$ the Fluxion of ED $\left(\text{where } d=\frac{t^2-c^2}{t^2}\right)$ in a ſeries is $\frac{t\dot{x}}{y}-\frac{d\dot{x}x^2}{2ty}-\frac{1d^2\dot{x}x^4}{2.4t^3y}-\frac{1.3d^3\dot{x}x^6}{2.4.6t^5y}-\frac{1.3.5d^4\dot{x}x^8}{2.4.6.8t^7y}$ &c. $=\dot{z}$, whoſe Fluent is thus found.

Aſſume $\dot{A}+\dot{B}+\dot{C}+\dot{D}+\dot{E}$ &c. $=\dot{z}$.

And let	Then by Fluxions
$x\sqrt{t^2-x^2}=xy=p$	$\frac{-\dot{x}x^2}{y}=\frac{\dot{p}}{2}-\frac{1t^2\dot{x}}{2y}$
$x^3\sqrt{t^2-x^2}=x^3y=q$	$\frac{-\dot{x}x^4}{y}=\frac{\dot{q}}{4}-\frac{3t^2\dot{x}x^2}{4y}$
$x^5\sqrt{t^2-x^2}=x^5y=r$	$\frac{-\dot{x}x^6}{y}=\frac{\dot{r}}{6}-\frac{5t^2\dot{x}x^4}{6y}$
$x^7\sqrt{t^2-x^2}=x^7y=s$	$\frac{-\dot{x}x^8}{y}=\frac{\dot{s}}{8}-\frac{7t^2\dot{x}x^6}{8y}$
$x^9\sqrt{t^2-x^2}=x^9y=t$	$\frac{-\dot{x}x^{10}}{y}=\frac{\dot{t}}{10}-\frac{9t^2\dot{x}x^8}{10y}$
&c.	&c.

Hence we have

$$\dot{A}=\frac{t\dot{x}}{y}.$$

$$\dot{B}=\frac{d\dot{x}x^2}{2ty}=\frac{d\dot{p}}{2.2t}-\frac{1dt\dot{x}}{2.2y}=\frac{d\dot{p}}{2.2t}-\frac{d\dot{A}}{2.2}.$$

Ċ =

$$\dot{C} = \frac{1d^2\dot{x}x^4}{2.4t^3y} = \frac{1d^2\dot{q}}{2.4.4t^3} - \frac{1.3d^2\dot{x}x^2}{2.4.4ty} = \frac{1d^2\dot{q}}{2.4.4t^3} + \frac{1.3d\dot{B}}{4.4}.$$

$$\dot{D} = \frac{1.3d^3\dot{x}x^6}{2.4.6t^5y} = \frac{1.3d^3\dot{r}}{2.4.6.6t^5} - \frac{1.3.5d^3\dot{x}x^4}{2.4.6.6t^3y} = \frac{1.3d^3\dot{r}}{2.4.6.6t^5} + \frac{3.5d\dot{C}}{6.6}.$$

$$\dot{E} = \frac{1.3.5d^4\dot{x}x^8}{2.4.6.8t^7y} = \frac{1.3.5d^4\dot{s}}{2.4.6.8.8t^7} - \frac{1.3.5.7d^4\dot{x}x^6}{2.4.6.8.8t^5y}$$
$$= \frac{1.3.5d^4\dot{s}}{2.4.6.8.8t^7} + \frac{5.7\dot{D}}{8.8}.$$

&c.

The Fluents of which are;	Or by reſtoring xy for p, x^3y for q, &c.
$A = \text{Fl.}\ \frac{t\dot{x}}{y}$	$A = \text{Fl.}\ \frac{t\dot{x}}{y}$
$B = \frac{dp}{2.2t} - \frac{dA}{2.2}$	$B = \frac{dxy}{2.2t} - \frac{dA}{2.2}$
$C = \frac{1d^2q}{2.4.4t^3} + \frac{1.3dB}{4.4}$	$C = \frac{1d^2x^3y}{2.4.4t^3} + \frac{1.3dB}{4.4}$
$D = \frac{1.3d^3r}{2.4.6.6t^5} + \frac{3.5dC}{6.6}$	$D = \frac{1.3d^3x^5y}{2.4.6.6t^5} + \frac{3.5dC}{6.6}$
$E = \frac{1.3.5d^4s}{2.4.6.8.8t^7} + \frac{5.7dD}{8.8}$	$E = \frac{1.3.5d^4x^7y}{2.4.6.8.8t^7} + \frac{5.7dD}{8.8}$
&c.	&c.

Which may be expreſſed thus;

$$A = \text{Fluent}\ \frac{t\dot{x}}{y}$$

$$B = P - \frac{dA}{2.2}$$

C =

$$C = \frac{1.2dx^2P}{4.4t^2} + \frac{1.3dB}{4.4}$$

$$D = \frac{3.4dx^2Q}{6.6t^2} + \frac{3.5dC}{6.6}$$

$$E = \frac{5.6dx^2R}{8.8t^2} + \frac{5.7dD}{8.8}$$

&c.

That is, $z = A, + \frac{dx\sqrt{t^2 - x^2}}{2.2t} - \frac{dA}{2.2} + \frac{1.2dx^2P}{4.4t^2} + \frac{1.3dB}{4.4} + \frac{3.4dx^2Q}{6.6t^2} + \frac{3.5dC}{6.6} + \frac{5.6dx^2R}{8.8t^2} + \frac{5.7dD}{8.8} + \frac{7.8dx^2S}{10\,10t^2} + \frac{7.9dE}{10\,10}$ &c. $=$ arch DE.

Note, A is put for the arch of a circle whoſe radius $= t$, and right ſine $= x$.

B for $\frac{dx\sqrt{t^2 - x^2}}{2.2t} - \frac{dA}{2.2}$

C for $\frac{1.2dx^2P}{4.4t^2} + \frac{1.3dB}{4.4}$

$D = \frac{3.4dx^2Q}{6.6t^2} + \frac{3.5dC}{6.6}$, &c.

And $P = \frac{dx\sqrt{t^2 - x^2}}{2.2t}$; $Q = \frac{1.2dx^2P}{4.4t^2}$, &c.

But when x flows $= t$, then $P = 0$, and therefore Q, R, &c. $= 0$, and DE becomes the quadrantal arch DA; $= A - \frac{dA}{2.2} + \frac{1.3dB}{4.4} + \frac{3.5dC}{6.6} + \frac{5.7dD}{8.8} + \frac{7.9dE}{10.10}$ &c. Where A, B, C, &c. are the preceding terms with their ſigns, and A is then the quadrantal arch of a circle whoſe radius is t; or if A be put for the periphery of the whole circle whoſe diameter is $2t$, then will the above *Theorem* at once exhibit the periphery of the whole ellipſis, and agrees exactly with Mr. *Cunn*'s.

To find the complemental arch EA, for which put u; then becauſe the equation of the curve is $\frac{t}{c}\sqrt{c^2-y^2}=x$, we have $\dot{u}=\frac{\dot{y}\sqrt{c^2+by^2}}{\sqrt{c^2-y^2}}$, (where $b=\frac{t^2-c^2}{c^2}$) whoſe Fluent may be found as above, Fig. 30.

and is $A-\frac{by\sqrt{c^2-y^2}}{2.2c}+\frac{bA}{2.2}, -\frac{1.2by^2P}{4.4c^2}+\frac{1.3bB}{4.4}, -\frac{3.4by^2Q}{6.6c^2}+\frac{3.5bC}{6.6}, -\frac{5.6by^2R}{8.8c^2}+\frac{5.7bD}{8.8}$ $-\frac{7.8by^2S}{10.10c^2}+\frac{7.9bE}{10.10}$ &c. $=u=$ AE, where A is put for the arch of that circle whoſe radius is c, and right ſine $=y$;

B for $-\frac{by\sqrt{c^2-y^2}}{2.2c}+\frac{bA}{2.2}$

C for $-\frac{1.2by^2P}{4.4c^2}+\frac{1.3bB}{4.4}$, &c.

And P for $-\frac{by\sqrt{c^2-y^2}}{2.2c}$; and Q for $-\frac{1.2by^2P}{4.4c^2}$, &c.

Example in Numbers.

Let $t=50$, $c=35$, what is the periphery of ſuch an ellipſis?

Here by this *Theorem* $A-\frac{dA}{2.2}+\frac{1.3dB}{4.4}+\frac{3.5dC}{6.6}+\frac{5.7dD}{8.8}$ &c. the periphery at once is found to be 269.118456 +.

Now let us ſee how the *Theorems* for finding the arcs DE and EA agree with this; by the firſt making $x=40$, DE will be found $=43.34399$ nearly; and by the laſt AE will be found $=24.111811$ nearly; but $43.34399+24.111811=$ elliptic quadrant, viz. $=67.455801$, therefore

$67.455801 \times 4 = 269.8232$, nearly agreeing with the other.

PROB. XI.

To find the area of curves.

If the abſciſſa of any curve be called x, its correſponding ordinate y, then where the ordinates are parallel, the Fluxion of the area is $y\dot{x}$.

Where the ordinates reſpect a point; let $r =$ radius of a circle deſcribed on that point, $x =$ abſciſſa, and $y =$ that part intercepted between the point and curve, then the Fluxion of the area is $\frac{y^2\dot{x}}{r}$, and by ſubſtituting in the room of $\dot{x}$ or y their values from the equation of the curve, there will ariſe an equation, whoſe Fluent is the area ſought.

Thus, if $y = x^n$, then $y\dot{x} = \dot{x}x^n$, whoſe Fluent is $yx = \frac{x^{n+1}}{n+1} = \frac{xx^n}{n+1} = \frac{1}{n+1}xy$. And in the *aſſymptotic ſpace* in the hyperbola, n being there negative, is $= - \frac{1}{n+1}xy$. Theſe things being premiſed concerning the quadrature of curves, to find the area, this is the

RULE.

1. Find the value of the ordinate by the help of the equation, expreſſing the nature of the curve.

2. Multiply this value by the Fluxion of the abſciſſa, the product ariſing therefrom is the Fluxion of the area.

3. Find the Fluent of that Fluxion, it gives the area ſought.

EXAM. I.

Given $aay = ax^2 - x^3$ *an equation to a curve, to find the area.*

1ſt, $y = \frac{ax^2 - x^3}{aa}$, which multiplied by $\dot{x}$ is $y\dot{x} = \frac{ax^2\dot{x} - x^3\dot{x}}{aa}$ the Fluxion of the area, whoſe Fluent is $\frac{ax^3}{3aa} - \frac{x^4}{4aa}$ the area; and when x by flowing becomes $= a$, it is $\frac{a^4}{3aa} - \frac{a^4}{4aa} = \frac{aa}{3} - \frac{aa}{4} = \frac{4a^2 - 3a^2}{12} = \frac{aa}{12}$.

EXAM. 2.

Given $ya^2 - a^3 = x^2y$ an equation to a curve, to find its area.

1. By tranſpoſition $ya^2 - x^2y = a^3$, then $y = \frac{a^3}{a^2 - x^2}$, which being multiplied by $\dot{x}$, we have $y\dot{x} = \frac{a^3\dot{x}}{a^2 - x^2}$ for the Fluxion of the area; by Caſe 4, Form 6, the Fluent is $= \frac{2.30258a^3}{2a} \times \log. \frac{a+x}{a-x}$, $= \frac{2.30258aa}{2} \times \log. \frac{a+x}{a-x}$ the area required.

EXAM. 3.

Let the nature of the curve be $y = x^5 - x^3 - x$, to find the area.

1. $y = x^5 - x^3 - x$, which multiply by $\dot{x}$, is $y\dot{x} = x^5\dot{x} - x^3\dot{x} - x\dot{x}$ the Fluxion of the area, whoſe Fluent is $yx = \frac{x^6}{6} - \frac{x^4}{4} - \frac{x^2}{2}$ the area ſought.

EXAM. 4.

Given $2ay + y^2 = 2ax$ an equation to a curve, to find the area.

By compleating the ſquare $y^2 + 2ay + aa = 2ax + aa$, whence $y = \sqrt{2ax + aa} - a$, which multi-

plied

plied by $\dot{x}$, is $y\dot{x} = \dot{x}\sqrt{aa+2ax} - a\dot{x}$, the Fluxion of the area. By Caſe 9, Form 3, the Fluent is $yx = \frac{\overline{aa+2ax}|^{\frac{3}{2}}}{\frac{3}{2}\times 2a} - ax = \frac{\overline{aa+2ax}|^{\frac{3}{2}}}{3a} - ax$ the area.

Suppoſe the curve was aſcertained by the nature of the Problem, and to ſee whether the above be the true Fluent, or whether any thing is to be added to, or ſubtracted from it; put $y=0$, then $x=0$, and by Prop. XXIII. $\frac{aa}{3}$ is to be taken from it; therefore the Fluent corrected is $= \frac{\overline{a^2+2ax}|^{\frac{3}{2}}}{3a} - ax - \frac{aa}{3}$ the area required.

But I ſhall content myſelf for the moſt part to give the Fluent in general.

Exam. 5.

Suppoſe the equation expreſſing the nature of a curve be $x^4 + a^2x^2 = yy$, *to find the area.*

1. $y = \sqrt{x^4+a^2x^2} = x\sqrt{x^2+a^2}$, which multiplied by $\dot{x}$ is $\dot{x}y = x\dot{x}\sqrt{x^2+a^2}$ the Fluxion of the area $= \overline{a^2+x^2}|^{\frac{1}{2}} \times x\dot{x}$.

Comparing this with Caſe 10, Form 3, we have $\alpha = a^2$, $\beta = 1$, $z = x$, $\mu = \frac{1}{2}$, then $\frac{\overline{\alpha+\beta z^2}|^{\mu+1}}{\mu+1.2\beta} = \frac{\overline{aa+xx}|^{\frac{3}{2}}}{\frac{3}{2}.2} = \frac{\overline{aa+xx}|^{\frac{3}{2}}}{3}$ the Fluent or area required.

Or thus.

Put $\sqrt{aa+xx} = z$, then $zz = aa+xx$, in Fluxions $2z\dot{z} = 2x\dot{x}$ or $z\dot{z} = x\dot{x}$, now ſubſtitute $z\dot{z}$ for $x\dot{x}$, and z for $\sqrt{aa+xx}$, the Fluxion thus freed from ſurds will be $= z^2\dot{z}$, whoſe Fluent is $\frac{z^3}{3}$, now reſtoring the value of z, it is $\frac{z^3}{3} = \frac{\overline{aa+xx}}{3}\sqrt{aa+xx}$ $= \frac{\overline{aa+xx}|^{\frac{3}{2}}}{3}$ as above.

EXAM. 6.

Let the equation to a curve be $ay^2 + xy^2 = x^2$, *to find the area.*

1. $y^2 = \frac{x^2}{a+x}$, and $y = \frac{x}{\sqrt{a+x}}$, multiplied by $\dot{x}$ is $y\dot{x} = \frac{x\dot{x}}{\sqrt{a+x}}$, the Fluxion of the area, whoſe Fluent by Form the 9th is $= \frac{2,30258}{2} \times \log. \sqrt{x} + \sqrt{a+x}$, the area required.

Or thus.

Put $z = \sqrt{a+x}$, then $zz = a + x$, and $x = zz - a$, in Fluxions $\dot{x} = 2z\dot{z}$, therefore $\frac{x\dot{x}}{\sqrt{a+x}} =$ $2z^2\dot{z} - 2a\dot{z}$, whoſe Fluent is $\frac{2z^3}{3} - 2az$, and putting $\sqrt{a+x}$ for z, then $z^3 = \overline{a+x}\sqrt{a+x}$, therefore the Fluent of $\frac{x\dot{x}}{\sqrt{a+x}} = \frac{2\overline{a+x}\sqrt{a+x}}{3} - 2a$ $\sqrt{a+x}$, the area expreſſed in ſurds.

EXAM. 7.

To find the area of a curve whoſe equation is $a\sqrt{2ax+x^2} + a\sqrt{2ax-x^2} = y\sqrt{4a^2x^2-x^4}$.

The given equation is $a\sqrt{2ax+xx} + a\sqrt{2ax-xx}$ $= y\sqrt{4a^2x^2-x^4}$.

By Diviſion.

$y = \frac{a\sqrt{2ax+xx}+a\sqrt{2ax-xx}}{\sqrt{4aaxx-x^4}} = \frac{a\sqrt{2ax+xx}}{\sqrt{4a^2x^2-x^4}} + \frac{a\sqrt{2ax-xx}}{\sqrt{4a^2x^2+x^4}}$. But $\frac{a\sqrt{2ax+xx}}{\sqrt{4a^2x^2-x^4}} = \frac{a}{\sqrt{2ax-xx}}$, and $\frac{a\sqrt{2ax-xx}}{\sqrt{4a^2x^2-x^4}} = \frac{a}{\sqrt{2ax+xx}}$, therefore $y = \frac{a}{\sqrt{2ax-xx}} + \frac{a}{\sqrt{2ax+xx}}$, and therefore the Fluxion of the area is $y\dot{x} = \frac{a\dot{x}}{\sqrt{2ax-xx}} + \frac{a\dot{x}}{\sqrt{2ax+xx}} = \frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}} + \frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a+x}}$. By Caſe 3, Form 9, the Fluent of $\frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a+x}} = 2.30258a \times 2 \log. \sqrt{x} + \sqrt{2a+x}$; and, by Caſe 3, Form 10, the Fluent of $\frac{ax^{-\frac{1}{2}}\dot{x}}{\sqrt{2a-x}}$ is $= .017453a \times$ deg. in arch whoſe ſine $= \frac{2}{2a}\sqrt{2ax-xx}$, radius $= 1$. Therefore $.017453a \times$ deg. in arch whoſe ſine $= \frac{2}{2a}\sqrt{2ax-xx}$, $+ 2.30258a \times 2 \log. \sqrt{x} + \sqrt{2a+x}$ is the area ſought.

EXAM. 8.

Required the quadrature of the curve whoſe equation is $y^3 = ax^2 + bx^{\frac{1}{2}}y$.

Aſſume $\frac{x^n}{v} = y$, then the curve's equation will take on it this form, $\frac{x^{3n}}{v^3} = ax^2 + \frac{bx^{n+\frac{1}{2}}}{v}$; make

$3n = n + \frac{1}{2}$, therefore $n = \frac{1}{4}$, and the above equation becomes $\frac{x^{\frac{3}{4}}}{v^3} = ax^2 + \frac{bx^{\frac{3}{4}}}{v}$; by reduction, it is $ax^{\frac{5}{4}} = \frac{1}{v^3} - \frac{b}{v}$, whoſe Fluxion is $\frac{5}{4}a\dot{x}x^{\frac{1}{4}} = \frac{b\dot{v}}{v^2} - \frac{3\dot{v}}{v^4}$, which divided by $\frac{5av}{4}$, gives $\frac{\dot{x}x^{\frac{1}{4}}}{v} = (y\dot{x} = \dot{u} =)\ \frac{4b\dot{v}}{5av^3} - \frac{12\dot{v}}{5av^5}$; whoſe Fluent is $u = \frac{3}{5av^4} - \frac{2b}{5av^2} + Q$, ſome conſtant quantity, but when $u = 0$, $x = 0$, and therefore $(ax^{\frac{5}{4}}) = 0 = \frac{1}{v^3} - \frac{b}{v}$, therefore $\frac{1}{v^2} = b$, which value being ſubſtituted in the above Fluent, we have $Q = \frac{-bb}{5a}$, therefore the correct Fluent is $\frac{3}{5av^4} - \frac{2b}{5av^2} - \frac{bb}{5a}$, and by reſtoring x and y we get $\frac{3y^4}{5ax} - \frac{2by^2}{5a\sqrt{x}} - \frac{bb}{5a} = u$, the area.

EXAM. 9.

Given $xx + yy = a\sqrt{xy}$ an equation to a curve, to find the area.

Put $y = \frac{x^3}{zz}$, ſquared is $yy = \frac{x^6}{z^4}$, now for yy put its equal, then the equation becomes $xx + \frac{x^6}{z^4} = \frac{axx}{z}$, and $x^2z^5 + x^6z = ax^2z^4$, dividing by x^2z, it is $z^4 + x^4 = az^3$, and by tranſpoſition $x^4 = az^3 - z^4$,

$-z^4$, which in Fluxions is $4x^3\dot{x} = 3az^2\dot{z} - 4z^3\dot{z}$, whence $\dot{x} = \frac{3az^2\dot{z} - 4z^3\dot{z}}{4x^3}$, but $y = \frac{x^3}{zz}$, which multiplied by $\dot{x}$ is $y\dot{x} = \frac{x^3}{zz} \times \frac{\overline{3az^2\dot{z} - 4z^3\dot{z}}}{4x^3} = \frac{3ax^3z^2\dot{z} - 4x^3z^3\dot{z}}{4x^3z^2} = \frac{3a\dot{z}}{4} - z\dot{z}$, the Fluxion of the area, whoſe Fluent is $\frac{3az}{4} - \frac{zz}{2}$ the area, or reſtoring x and y is $\frac{3ax}{4}\sqrt{\frac{x}{y}} - \frac{x^3}{2y}$.

EXAM. 10.

Given $x^3 + y^3 = axy$, *to find the quadrature of the curve expreſſed by this equation.*

Put $y = \frac{xx}{z}$, which cubed is $y^3 = \frac{x^6}{z^3}$, for y^3 put its equal, then the equation becomes $x^3 + \frac{x^6}{z^3} = \frac{ax^3}{z}$, therefore $x^3z^4 + x^6z = ax^3z^3$, dividing by x^3z, the equation is $z^3 + x^3 = az^2$, whence $x^3 = az^2 - z^3$, in Fluxions is $3x^2\dot{x} = 2az\dot{z} - 3z^2z$, therefore $\dot{x} = \frac{2az\dot{z} - 3z^2\dot{z}}{3xx}$, but $y = \frac{xx}{z}$, then $y\dot{x} = \frac{xx}{z} \times \frac{\overline{2az\dot{z} - 3z^2\dot{z}}}{3} = \frac{2ax^2z\dot{z} - 3x^2z^2\dot{z}}{3zxx} = \frac{2a\dot{z}}{3} - z\dot{z}$, whoſe Fluent is $\frac{2az}{3} - \frac{zz}{2}$ the area, but $z = \frac{xx}{y}$, then reſtoring x and y, the area is $\frac{2axx}{3y} - \frac{x^4}{2yy}$.

EXAM. 11.

To find the area of a curve whoſe equation is $x^4 + y^4 = axy$.

Put $y = vx$, then $y^4 = v^4x^4$, and by ſubſtituting v^4x^4 for y^4, the equation becomes $x^4 + v^4x^4 = ax^2v$; dividing by xx, is $xx + xxv^4 = av$, whence $xx = \frac{av}{1+v^4}$; in Fluxions is $2x\dot{x} = \frac{a\dot{v}}{1+v^4} - \frac{4av^4\dot{v}}{\overline{1+v^4}|^2}$, but $y\dot{x} = vx\dot{x} = \frac{\frac{1}{2}av\dot{v}}{1+v^4} - \frac{2av^5\dot{v}}{\overline{1+v^4}|^2}$, the Fluxion of the area, whoſe Fluent is (ſee *Ex.* 52.) $-\frac{1}{2}a\varphi + \frac{\frac{1}{2}avv}{1+v^4}$, or $-\frac{1}{2}a\varphi + \frac{\frac{1}{2}ay^2x^2}{y^4+x^4}$, where $\varphi = \frac{.017453}{2} \times$ degrees in the arch whoſe tangent is vv or $\frac{yy}{xx}$.

EXAM. 12. Univerſally.

If the relation between the abſciſſa and ordinate of a curve be expreſſed by $ay^nx^n = y^{2n+1} + x^{2n+1}$, *to find the area.*

Put $y = \frac{x^{\overline{\frac{n+1}{n}}}}{z}$, and ſubſtituting in the given equation, we ſhall have $x^{\overline{\frac{2n+1}{n}}} = az^{n+1} - z^{2n+1}$, and conſequently $y\dot{x} = \frac{a \times \overline{n+1} \times n}{2n+1} z^{n-1}\dot{z} - nz^{2n-1}\dot{z}$, whoſe Fluent is $\frac{a \times \overline{n+1}}{2n+1} z^n - \frac{1}{2}z^{2n}$, and by reſtoring $\frac{x^{n+1}}{y^n}$ for z^n, and $\frac{x^{2n+2}}{y^{2n}}$ for z^{2n}, we have $\frac{a \times \overline{n+1}}{2n+1} \times \frac{x^{n+1}}{y^n} - \frac{x^{2n+2}}{2y^{2n}}$ for the area, which, by expunging x^{2n+1}, becomes $\frac{yx}{2} + \frac{ax^{n+1}}{4n+2 \cdot y^{n}}$.

1. Let $n = \frac{1}{2}$, then the equation of the curve will be $x^2 + y^2 = a\sqrt{xy}$, and the area without correction is $\frac{3ax}{4}\sqrt{\frac{x}{y}} - \frac{x^3}{2y}$, as above, or $\frac{yx}{2} + \frac{ax^{\frac{3}{2}}}{4y^{\frac{1}{2}}}$ the correct area.

2. Let $n = 1$, the equation of the curve will be $x^3 + y^3 = axy$, and the area will be $\frac{2axx}{3y} - \frac{x^4}{2yy}$, the ſame as above, or it is $\frac{yx}{2} + \frac{axx}{6y}$ the correct area.

EXAM. 13.

Suppoſe the equation of a curve is $y^3 + ax = x^2y$, *to find the area.*

Aſſume $\frac{x^n}{v} = y$, then the equation becomes $\frac{x^{3n}}{v^3} + ax = \frac{x^{n+2}}{v}$. Therefore if $3n = n + 2$, then $n = 1$, and the equation is $\frac{x^3}{v^3} + ax = \frac{x^3}{v}$, and by reduction $x^2 = \frac{av^3}{v^2 - 1}$, whoſe Fluxion is $2x\dot{x} = \frac{av^4\dot{v} - 3av^2\dot{v}}{\overline{vv-1}|^2}$. Therefore $\frac{x\dot{x}}{v} = y\dot{x}$ the Fluxion of the area $= \frac{\frac{1}{2}av^3\dot{v} - \frac{3}{2}av\dot{v}}{\overline{vv-1}|^2}$, whoſe Fluent is $\frac{a}{4}$

× 2.30258

$\times\ 2.30258 \text{ log. } \overline{vv-1} + \frac{a}{2.\overline{vv-1}} + Q$ ſome conſtant quantity for correction.

If $3n = 1$, then $n = \frac{1}{3}$, and the equation of the curve becomes $\frac{x}{v^3} + ax = \frac{x^{\frac{7}{3}}}{v}$, therefore $x^{\frac{4}{3}} = av + \frac{1}{vv}$, whoſe Fluxion is $\frac{4x^{\frac{1}{3}}\dot{x}}{3} = a\dot{v} - \frac{2\dot{v}}{v^3}$, which equation multiplied by $\frac{3}{4v}$, gives $\frac{x^{\frac{1}{3}}\dot{x}}{v} = \frac{3a\dot{v}}{4v} - \frac{3\dot{v}}{2v^4} = y\dot{x}$, the Fluxion of the area whoſe Fluent is $\frac{1}{2v^3} + \frac{3}{4}a \times 2.30258 \text{ log. } v - Q$, ſome conſtant quantity for correction.

If $n + 2 = 1$, then $n = -1$, and the equation becomes $\frac{1}{v^3x^3} + ax = \frac{x}{v}$, therefore $x^4 = \frac{1}{vv - av^3}$, or $-\frac{1}{x^4} = av^3 - v^2$, whoſe Fluxion is $\frac{4\dot{x}}{x^5} = 3av^2\dot{v} - 2v\dot{v}$, and multiplying theſe two laſt equations together we have $\frac{4\dot{x}}{x} = \frac{3av^2\dot{v} - 2v\dot{v}}{vv - av^3}$. Therefore $\frac{4\dot{x}}{vx} = 4y\dot{x} = \frac{3av\dot{v} - 2\dot{v}}{vv - av^3}$, whoſe Fluent is F. $yx = 2.30258a \times \overline{\text{log. } v} + \frac{2}{v} + a^2v + \frac{a^3v^2}{2} + \frac{a^4v^3}{3} + \frac{a^5v^4}{4}$ &c. $+ Q$, ſome conſtant quantity $=$ four times the area.

Or the Fluent of $\frac{3av\dot{v} - 2\dot{v}}{vv - av^3}$ may be found by Mr. *Emerſon*'s Table.

Thus, $\frac{3av\dot{v} - 2\dot{v}}{vv - qv^3}$ is $= \frac{av\dot{v} + 2av\dot{v} - 2\dot{v}}{vv - av^3} =$ $\frac{av\dot{v}}{v^2 - av^3} - \frac{2\dot{v}}{vv} = \frac{av^{-2}\dot{v}}{-a + v^{-1}} - 2v^{-2}\dot{v}$, whoſe Fluent by Form 4 and 1 is $= \frac{2.30258}{-1} \times \log. \overline{-a+v^{-1}}; + \frac{2}{v} = \frac{2}{v} - 2.30258 \times \log. \frac{1-av}{v}$.

EXAM. 14.

Let $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$ be an equation to a curve, to find the area.

Aſſume $\frac{v\dot{y}}{a} = \dot{x}$, then the equation becomes $\frac{vy\dot{y}^4}{a} = \frac{v^4\dot{y}^4}{a^3} + \frac{2v^2\dot{y}^4}{a} + a\dot{y}^4$; multiplying both ſides by $\frac{a^3}{\dot{y}^4}$, we ſhall have $a^2vy = v^4 + 2a^2v^2 + a^4$, whence $y = \frac{v^3}{a^2} + 2v + \frac{a^2}{v}$; in Fluxions $\dot{y} = \frac{3v^2\dot{v}}{a} + 2\dot{v} - \frac{a^2\dot{v}}{v^2}$. Therefore $\frac{v\dot{y}}{a}$, or $\dot{x} = \frac{3v^3\dot{v}}{a^3} + \frac{2v\dot{v}}{a} - \frac{a\dot{v}}{v}$; and the Fluxion of the area $y\dot{x} = \frac{3v^6\dot{v}}{a^5} + \frac{8v^4\dot{v}}{a^3} + \frac{6v^2\dot{v}}{a} - \frac{a^3\dot{v}}{v^2}$, the Fluent of which is $\frac{3v^7}{7a^5} + \frac{8v^5}{5a^3} + \frac{2v^3}{a} + \frac{a^3}{v}$, the area required.

A general propoſition of the binomial or reſidual form, with its canon, for finding the area of all ſuch figures, as may be expreſſed in finite terms.

PROB.

PROB. XII. A PROP:

If in any plane figure, whoſe abſciſſa is $= x$, *and its perpendicular ordinate* $= y$, *the relation of* x *to* y *be* $y = dx^{tn-1} \times \overline{c+bx^{n}}|^{r}$; d, c, b, *and the ſuperiors* t, n, *and* r *being any given quantities, ſo as* n *be not* $\sqsupset 0$, *and* x *and* y *variable ones, it is required to find the area* (A) *of ſuch figure, that is, to find the Fluent.*

CANON.

Make $s = r + t$, $h = \frac{d}{b^{t}n}$, and $B = c + bx^{n}$,

then A the area ſought is $= \frac{h}{s} \times : B^{s} - c^{s} : -$

$$\frac{t-1}{1} \times \frac{hc}{s-1} \times : B^{s-1} - c^{s-1} : + \frac{t-1}{1} \times \frac{t-2}{2}$$

$$\times \frac{hc^{2}}{s-2} \times : B^{s-2} - c^{s-2} : - \frac{t-1}{1} \times \frac{t-2}{2} \times$$

$$\frac{t-3}{3} \times \frac{hc^{3}}{s-3} \times : B^{s-3} - c^{s-3} : + \frac{t-1}{1} \times \frac{t-2}{2}$$

$$\times \frac{t-3}{3} \times \frac{t-4}{4} \times \frac{hc^{4}}{s-4} \times : B^{s-4} - c^{s-4} \text{ \&c.}$$

Note. When the ordinate (*viz.* the value of y) is deſigned ſo as one of the two terms under the *vinculum* is an invariable given quantity, and the index $= n$ of x in the other term is not $\sqsupset 0$; then is the ordinate expreſſed in a proper form for the ſolution. And then,

When it is $=$ any integer number not $\sqsupset 0$, t, $t-1$, $t-2$, or $t-3$, &c. will be $= 0$; by which means the value of A ſought will be found in finite terms by the above canon; as will farther appear by the following Examples.

EXAM.

EXAM. I.

If $y = \dfrac{dx}{l^{2}-2lx^{2}+x^{4}}$, *quere* A.

The value of y being reduced to the form required by our propoſition is $= dx \times \overline{l-x^{2}}|^{-2}$; wherein $n = 2$, and therefore $\sqsupset 0$, and $tn - 1$ $(= 2n - 1) = 1$, therefore $t = 1$, which is an integer number not leſs than 0; conſequently the value of A may be had in finite terms.

Thus $d = d$, $c = l$, $b = -1$, and $r = -2$; wherefore s $(= r + t) = -1$; b $\left(= \dfrac{d}{b^{t}n}\right) = \dfrac{d}{-1^{1}\times 2}$; and $B = l - x^{x}$. And then A $\left(= \dfrac{d}{-2\times -1} \times : \overline{l-x^{2}}|^{-1} - l^{-1} : - 0\right)$ is $= \dfrac{dx^{2}}{2ll-2lx^{2}}$.

Now let us ſee what the reſult is of making n leſs than 0, by putting $y = dx^{-3} \times \overline{-1+lx^{-2}}|^{-2}$, which is called the ſecond form.

Here $n = -2$, and $tn - 1$ $(= -2t - 1) = -3$, conſequently $t = 1$, which is an integer number not leſs than 0.

Then $d = d$, $c = -1$, $b = l$, and $r = -2$. Wherefore s $(= r + t) = -1$, b $\left(= \dfrac{d}{l^{t}\times -2}\right) = \dfrac{d}{-2l}$; and $B = -1 + lx^{-2}$: and thus A $\left(= \dfrac{d}{2l\times -1} \times : \overline{-1+lx^{-2}}|^{-1} + t^{-1} :\right)$ would be $\dfrac{d}{2l-2xx}$.

Now this being not the ſame with the former, cannot be the true area: though in this example (and it happens ſo in ſome other examples) it is what I call the imperfect area, *viz.* the Fluent of this figure. For by putting $x = 0$, the reſult of this Fluent is $\frac{d}{2l}$; from which ſubtract the ſaid Fluent, and the remainder is $= \frac{dxx}{2ll - 2lxx} = A$, as at firſt.

EXAM. 2.

If y be $= \frac{dx^{\frac{1}{3}}}{\sqrt{f^3 - 3f^2gx^{\frac{2}{3}} + 3fg^2x^{\frac{4}{3}} - g^3x^2}}$, *quere* A.

This value of y being reduced to the form required by our propoſition is $= dx^{\frac{1}{3}} \times \overline{f - gx^{\frac{2}{3}}}|^{-\frac{3}{5}}$, wherein n is $= \frac{2}{3}$, and therefore not leſs than 0, and $tn - 1$ $(= \frac{2}{3}t - 1) = \frac{1}{3}$; therefore t is $= 2$, which is an affirmative integer number; conſequently the value of A may be had in finite terms.

Thus $d = d$, $c = f$, $b = -g$, and $r = -\frac{3}{5}$; wherefore s $(= r + t = -\frac{3}{5} + 2) = \frac{1}{5}$, b $\left(= \frac{d}{b^t n}\right)$ $= \frac{d}{g^2 \times \frac{2}{3}}$, and $B = f - gx^{\frac{2}{3}}$. And then

A $\left(= \frac{d}{\frac{2}{3}g^2 \times \frac{7}{5}} \times : \overline{f - gx^{\frac{2}{3}}}|^{\frac{7}{5}} - f^{\frac{7}{5}} : - \frac{1}{1} \times \frac{3df}{2g^2 \times \frac{2}{5}} \times \overline{f - gx^{\frac{2}{3}}}|^{\frac{2}{5}} - f^{\frac{2}{5}} :\right)$ is $= - \frac{30gx^{\frac{2}{3}} + 75f}{28gg} d \times \overline{f - gx^{\frac{2}{3}}}|^{\frac{2}{5}}$ $+ \frac{75d}{28gg} f^{\frac{7}{5}}$.

If it be objected that, though my canon doth not extend to the area itſelf, in purſuance of the ſaid 2d form; yet it ſeems, by what I have added in my firſt example to give, what I call the imperfect area, *viz.* the Fluent; which may be corrected by putting the fluxionary letter therein $= 0$, and then finding the difference of this reſult and the ſaid Fluent; for in either of the firſt or ſecond forms, Fig. forms, the area or Fluent ought to be corrected after it is found, and then they will come both to the ſame thing.

EXAM. 14.

To find the area of a triangle.

31. Put AB $= a$, CD $= b$, Cb $= x$, $cd = y$, then by ſimilar triangles $b : a :: x : y$, whence $by = ax$, and $y = \frac{ax}{b}$, therefore $\dot{y} = \frac{ax\dot{x}}{b}$ the Fluxion of the area, whoſe Fluent is $\frac{axx}{2b}$, for $\frac{ax}{b}$ ſubſtitute y its equal, then $\frac{axx}{2b}$ becomes $\frac{xy}{2}$ area of the ſpace Ccbd, and when b flows to D, cd will coincide with AB, then the area of the triangle ACB $= \frac{ab}{2}$.

EXAM. 15.

To find the area of a parabola.

Put AB $= a$, BC $= b$, AP $= x$, PM $= y$; the
25. equation to the parabola is $ax = yy$; in Fluxions is $\dot{x} = 2y\dot{y}$, which multiply by y gives $y\dot{x} = 2y^2\dot{y}$ the Fluxion of the area, whoſe Fluent is $y\dot{x} = \frac{2y^3}{3}$, and by ſubſtituting x for yy its equal in the expreſſion $\frac{2y^3}{3}$, then the area $= \frac{2}{3}xy$. Now when P flows to B, PM then becomes BC, and the area $= \frac{2}{3}ab$.

Univerſally.

Let $ax^m = y^n$ *be an equation for various parabolas,* then $y = a^{\frac{1}{n}} x^{\frac{m}{n}}$, which multiplied by $\dot{x}$ gives $y\dot{x} = a^{\frac{1}{n}} x^{\frac{m}{n}} \dot{x}$ the Fluxion of the area, whoſe Fluent is $\frac{n}{m \mp n}$

$\frac{n}{m+n} a^{\frac{1}{n}} x^{\frac{m}{n}+1}$, and by reſtoring y, by ſubſtituting its value above, is $\frac{n}{m+n}xy$ the area.

1.

Let $ax = y^2$. Here $m=1$, $n=2$; then $\frac{n}{m+n}xy$ is $= \frac{2}{3}xy$ the area.

2.

Let $ax = y^3$. Here $m=1$, $n=3$; then $\frac{n}{m+n}xy = \frac{3}{4}xy$ the area.

3.

Let $ax=y^4$. Here $m=1$, $n=4$; then $\frac{n}{m+n}xy$ $= \frac{4}{5}xy$ the area.

4.

Let $ax^2=y$. Here $m=2$, $n=1$; then $\frac{n}{m+n}xy$ $= \frac{1}{3}xy$ the area.

5.

Let $ax^3=y$. Here $m=3$, $n=1$; then $\frac{n}{m+n}xy$ $= \frac{1}{4}xy$ the area.

EXAM. 16.

To find the area of a circle.

Put CM = CD = r, PC = x, PM = y, then by *Theo.* 11, *Geo.* $xx+yy=rr$, and $rr-xx=yy$, and therefore $y=\sqrt{rr-xx}$, which multiply by $\dot{x}$, is $y\dot{x} = \dot{x}\sqrt{rr-xx}$ the Fluxion of the area. 32.

But $\sqrt{rr-xx} = r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5}$ &c. which multiply by $\dot{x}$, then $\dot{x}\sqrt{rr-xx} = r\dot{x} -$

Fig. 32. $\frac{x^2\dot{x}}{2r} - \frac{x^4\dot{x}}{8r^3} - \frac{x^6\dot{x}}{16r^5}$ &c. whoſe Fluent is $rx - \frac{x^3}{2\cdot 3r} - \frac{x^5}{5\cdot 8r^3} - \frac{x^7}{7\cdot 16r^5}$ &c. $=$ ſpace PMDC; but when PM flows to A, it then vaniſhes, and $x = r = 1 = AC$. And the ſeries becomes $1 - \frac{1}{2\cdot 3} - \frac{1}{5\cdot 8} - \frac{1}{7\cdot 16}$ &c. $=$ the quadrant ADC, which multiplied by 4 gives the area of the whole circle.

Or if the diameter $AB = 1$, then the ſeries will be the area of the whole circle $= .785398$.

Or thus.

The ſum of the ſeries $\frac{2y\sqrt{2y-yy}}{3} \times$ into $1 + \frac{y}{5x} - \frac{y^2}{5\cdot 7x^2} + \frac{3y^3}{5\cdot 7\cdot 9x^3} - \frac{3y^4}{7\cdot 9\cdot 11x^4} + \frac{3y^5}{9\cdot 11\cdot 13x^5}$ — &c. when $x = 2 - y$, and $y = 1$, expreſſes the area of the quadrant of a circle whoſe radius is 1.

The given ſeries is $\frac{2}{3} + \frac{1}{5}A - \frac{1}{7}B + \frac{3}{9}C - \frac{5}{11}D + \frac{7}{13}E$ &c. Compare this with *Prop.* VII. p. 35, of my edition of *Stirling*'s *Series*, and you will find $T^1 = \frac{z-3}{z\times 1}T$ for the equation of the terms, and the ſucceſſive values of z will be $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. and $n = 3$, $m = 2$; then

$$S = \tfrac{1}{2}T + \frac{3}{z} \times \frac{A}{2} + \frac{4}{z+1} \times \frac{B}{2} + \frac{5}{z+2} \times \frac{C}{2} + \text{\&c.}$$

Sum up five firſt terms of the given ſeries, which is .784416, and put $T = \frac{7}{13}E$, the 6th term, and the value of z (correſponding to T) will be $7\frac{1}{2}$, whence we have

S =

$$S = \frac{7}{2.13}E + \frac{3A}{15} + \frac{4B}{17} + \frac{5}{19}C + \frac{6}{21}D$$ Fig.

$$+ \frac{7}{23}E \text{ \&c.}$$

That is, $S = $ .000777
155
37
9
3
1

.000982
Add .784416

The ſum .785398 is the area of a quadrant of a circle whoſe radius is 1. Alſo ſee *Ex.* 3, p. 59.

Exam. 17.

To find the area of an ellipſis.

Put $AD = DB = t$, $CD = c$, $DP = x$, $PM = y$, 33. then $PB = t - x$, and $AP = t + x$; then by property of the ellipſis $\overline{AD}^2 : AP \times PB :: \overline{DC}^2 : \overline{PM}^2$, that is, $tt : tt - xx :: cc : yy$, therefore $ttyy = \overline{tt - xx} \times cc$ and $y = \frac{c}{t}\sqrt{tt - xx}$, which multiplied by $\dot{x}$ gives $y\dot{x} = \frac{c\dot{x}}{t}\sqrt{tt - xx}$, the Fluxion of the area of the part DCMP.

Now $\sqrt{tt - xx} = t - \frac{xx}{2t} - \frac{x^4}{8t^3} - \frac{x^6}{16t^5} - \frac{5x^8}{128t^7}$ &c. which multiplied by $\dot{x}$, is $t\dot{x} - \frac{x^2\dot{x}}{2t} - \frac{x^4\dot{x}}{8t^3} - \frac{x^6\dot{x}}{16t^5} - \frac{5x^8\dot{x}}{128t^7}$ &c. whoſe Fluent is $= tx - \frac{x^3}{2.3t} - \frac{x^5}{5.8t^3} - \frac{x^7}{7.16t^5} - \frac{5x^9}{9.128t^7}$ &c. which mul-

Fig. 33. tiplied by $\frac{c}{t}$ is $= cx - \frac{cx^3}{2.3tt} - \frac{cx^5}{5.8t^4} - \frac{cx^7}{7.16t^6} - \frac{5cx^9}{9.128t^8}$ &c. for the area DCMP. Now when P flows to B, then PM vaniſhes, and DP becomes $=$ DB, or $x = t$, and therefore the above ſeries will be $tc - \frac{tc}{2.3} - \frac{tc}{5.8} - \frac{tc}{7.16} - \frac{5tc}{9.128}$ &c. the area of the quadrant DCMB, which multiplied by 4 gives the area of the whole ellipſis, or $4tc \times$ $\frac{1}{2.3} - \frac{1}{5.8} - \frac{1}{7.16} - \frac{1}{9.128}$ &c.

PROB. XIII.

To find the ſurfaces of ſolids generated by the rotation of planes about an axis.

Put the abſciſſa $= x$, ordinate $= y$, the generating curve $= z$, $c = 3.1416 \times 2 =$ circumference of a circle whoſe radius is 1; S $=$ ſurface generated; then this

RULE.

Multiply the value of the circumference of the circle generated by the rotation of any ordinate about the axis of the curve into the Fluxion of the curve line, the product will give the Fluxion of the ſurface which is generated by the rotation of the curve; the Fluent thereof will give the ſurface; that is,

$\dot{s} = cy\dot{z}$ or $cy \times \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, for $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, *whoſe Fluent is* s *the ſurface required.*

EXAM. 1.

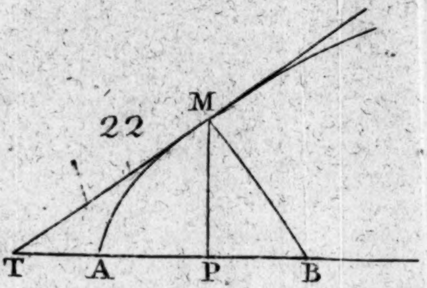
To find the ſurface of a pyramid.

34. Let ACB repreſent one of the faces of a pyramid ſtanding on a ſquare baſe, and put CD $= a$,

AB

1

22



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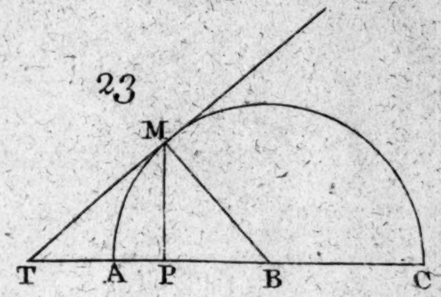

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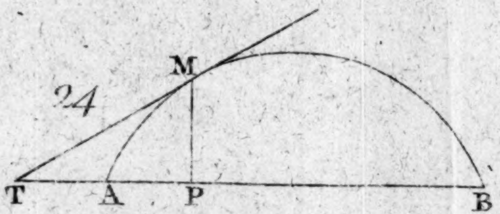

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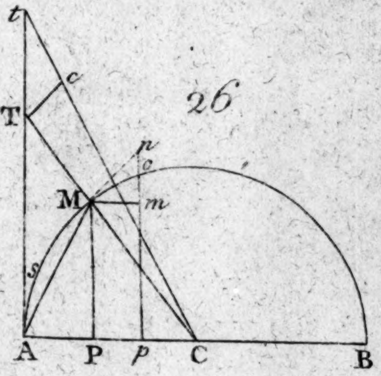

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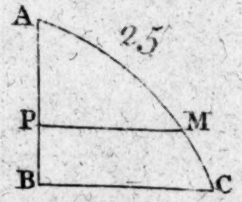

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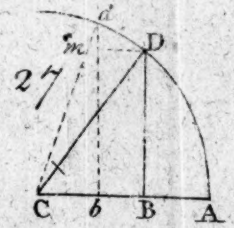

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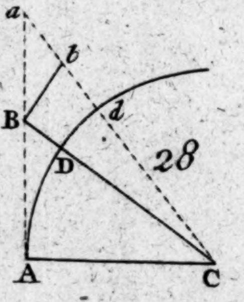

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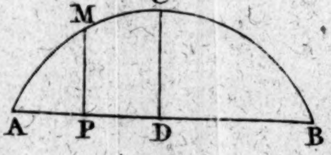

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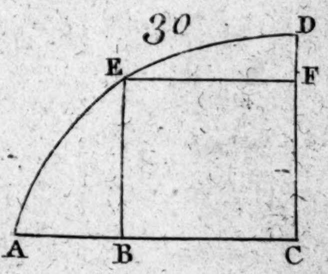

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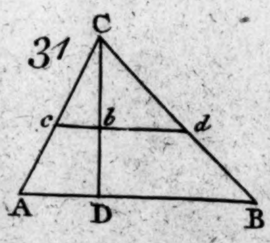

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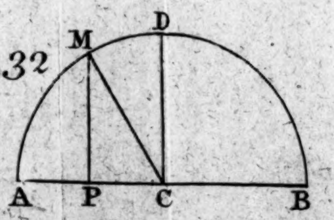

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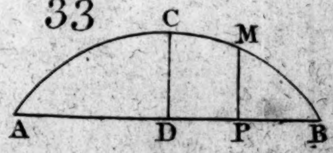


AB $= b$, C$d = x$, $ab = y$, then ſince the triangles Fig. Cab, and CAB are ſimilar, it will be $x : y :: a : b$, 34. then $ay = bx$, and $y = \frac{bx}{a}$, which multiply by $\dot{x}$, then $y\dot{x} = \frac{bx\dot{x}}{a}$ is the Fluxion of the area Cab, whoſe Fluent is $yx = \frac{bx^2}{2a}$; now put y for $\frac{bx}{a}$ its equal, then $\frac{bx^2}{2a}$ becomes $\frac{xy}{2}$ for the area of the ſpace Cab, and if the point d flow to D, then will ab coincide with AB, and x will be $= a$, and $y = b$, therefore $\frac{ab}{2} =$ area.

EXAM. 2.

To find the ſurface of a right cone,

Put CD $= a$, AD $= b$, C$d = x$, $ad = y$, curve 35. AC $= z$, then $cy\dot{z}$ is the Fluxion of the ſurface, but $\dot{z} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, then $cy\dot{z} = cy\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ is the Fluxion of the ſurface; but Cd : ad :: CD : AD; that is, $x : y :: a : b$, hence $bx = ay$, and $x = \frac{ay}{b}$, in Fluxions is $\dot{x} = \frac{a\dot{y}}{b}$, ſquared is $\dot{x}\dot{x} = \frac{a^2\dot{y}\dot{y}}{bb}$, now for $\dot{x}\dot{x}$ put its equal $\frac{a^2\dot{y}\dot{y}}{bb}$ in the expreſſion $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, then $cy\dot{z} = cy\sqrt{\frac{aa\dot{y}\dot{y}}{bb} + \dot{y}\dot{y}} = cy\sqrt{aa\dot{y}\dot{y} + bb\dot{y}\dot{y}} = \frac{cy\dot{y}}{bb}\sqrt{aa + bb}$, whoſe Fluent is $\frac{cy^2}{2bb}\sqrt{aa + bb}$, the ſurface of the cone aCb, but when d flows to D, then ad coincides with AD, and $y = b$; conſequently $\frac{c}{2}\sqrt{aa+bb} =$ ſurface of the whole cone ACB; but $\sqrt{aa+bb} = z =$ AC, therefore $\frac{cz}{2} =$ whole ſurface.

EXAM. 3.

To find the ſurface of a ſphere.

Let $a =$ radius, then by the property of the figure $yy = 2ax - xx$, in Fluxions $2y\dot{y} = 2a\dot{x} - 2x\dot{x}$, or $y\dot{y} = a\dot{x} - x\dot{x}$, whence $\dot{y} = \frac{a\dot{x} - x\dot{x}}{y}$, ſquared is $\dot{y}\dot{y} = \frac{a^2\dot{x}^2 - 2ax\dot{x}^2 + x^2\dot{x}^2}{yy}$, but $yy = 2ax - xx$, for which put, then $\dot{y}\dot{y} = \frac{a^2\dot{x}^2 - 2ax\dot{x}^2 + x^2\dot{x}^2}{2ax - xx}$; now $\dot{s} = cy\dot{z} = cy\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ the Fluxion of the ſurface; for $\dot{y}\dot{y}$ put its equal above, then we ſhall have

$$\dot{s} = cy\sqrt{\dot{x}\dot{x} + \frac{a^2\dot{x}^2 - 2ax\dot{x}^2 + x^2\dot{x}^2}{2ax - xx}} = cy$$

$$\sqrt{2ax\dot{x}^2 - x^2\dot{x}^2 + a^2\dot{x}^2 - 2ax\dot{x}^2 + x^2\dot{x}^2} = cy\sqrt{\frac{a^2\dot{x}^2}{2ax - xx}}$$

$$= cy \times \frac{\sqrt{a^2\dot{x}^2}}{\sqrt{2ax - xx}} = cy \times \frac{a\dot{x}}{y} = \frac{cay\dot{x}}{y} = ca\dot{x},$$

whoſe Fluent is $s = cax$ the curve ſurface of the ſemi-ſphere, and when $x = 2a$, then $4caa$, the ſurface of the whole ſphere.

EXAM. 4.

To find the curve-ſurface of a ſemi cubical paraboloid.

Let $x =$ abſciſſa, $y =$ ordinate of the generating parabola, then by nature of the figure $ax^2 = y^3$, or $ax^m = y^{m+1}$, whence $x = \frac{y^{\frac{m+1}{m}}}{a^{\frac{1}{m}}}$, in Fluxions

Fig.

$\dot x = \frac{m+1}{ma^{\frac{1}{m}}} y^{\frac{1}{m}} \dot y$; ſquared is $\dot x \dot x = \left.\overline{\frac{m+1}{ma^{\frac{1}{m}}}}\right|^{2} y^{\frac{2}{m}} \dot y^{2}$,

or putting $b = \left.\overline{\frac{m+1}{ma^{\frac{1}{m}}}}\right|^{2}$, we ſhall have $\sqrt{\dot x^{2} + \dot y^{2}} = \dot y$

$\sqrt{1 + by^{\frac{2}{m}}}$; but $\dot s = cy\dot y \sqrt{1 + by^{\frac{2}{m}}} = cy\dot y \sqrt{1+by}$ the Fluxion of the curve ſuperficies, $= cy\dot y \times$ $\overline{1 + \frac{by}{2} - \frac{b^{2}y^{2}}{8} + \frac{b^{3}y^{3}}{16} - \frac{b^{4}y^{4}}{32}}$ &c. whoſe Fluent is $\frac{cy^{2}}{2} + \frac{bcy^{3}}{6} - \frac{b^{2}cy^{4}}{32} + \frac{cb^{3}y^{5}}{80} - \frac{cb^{4}y^{6}}{192}$ &c. the curve ſurface of the whole paraboloid.

Let $y = 12$, then the curve ſurface is $= 582.5$.

PROB. XIV.

To find the content of ſolid bodies.

Multiply the area of the baſe deduced from the equation expreſſing the nature of the curve, by the Fluxion of the axis, the Fluent thereof is the ſolidity.

Let abſciſſa $= x$, ordinate $= y$, $c = 3.1416$, S = ſolid content, and $\dot S = cyy\dot x$ the Fluxion of the variable ſolidity.

Then by the equation of the curve exterminate one of the indetermined quantities, and its Fluxion, out of the above equation $\dot S = cyy\dot x$, and the Fluent gives the ſolidity.

EXAM. I.

To find the ſolidity of a pyramid.

Suppoſe the pyramid ſtand upon a ſquare baſe; 34 let AB the ſide of the baſe $= a$, CD $= b$, C$d = x$, Fig. 34. $ab = y$, then by ſimilar triangles we have $a : b :: y : x$, whence $ax = by$, and $x = \frac{by}{a}$, in Fluxions is $\dot{x} = \frac{b\dot{y}}{a}$, but $yy\dot{x} = \dot{S}$ the Fluxion of the variable ſolidity C*mn*, for $\dot{x}$ put $\frac{b\dot{y}}{a}$ its equal, then $yy\dot{x}$ becomes $\frac{by^2\dot{y}}{a} = \dot{S}$, whoſe Fluent is $\frac{by^3}{3a} = S$; now if we conceive C*d* in a flowing ſtate, and to move till it arrives at D, then *ab* coincides with AB, and y becomes $= a$, wherefore $\frac{ba^3}{3a} = \frac{ba^2}{3} = S$, the ſolidity of the pyramid ACB = area of the baſe multiplied by one third of the altitude.

EXAM. 2.

To find the ſolidity of a cylinder.

36. Let AE $= a$, EF $= b$, E$n = x$, $mn = y$, then $cyy\dot{x} = \dot{S}$, the Fluxion of the variable ſolidity AB*pnm*, whoſe Fluent is $S = cyyx$; now if we conceive x in a flowing ſtate, and to move till it arrives at F, then *mn* will coincide with CF, and y becomes $= a$, and $x = b$, therefore $S = caab$, the ſolidity of the whole cylinder.

Let AE $= 12 = a$, EF $= 60 = b$, then $caab = 27143.424$ the ſolidity.

EXAM. 3.

To find the ſolidity of a cone.

37. Put CD $= a$, AD $= b$, C$o = x$, $mo = y$, then $x : y :: a : b$, whence $ay = bx$, and $y = \frac{bx}{a}$, ſquared is $yy = \frac{b^2x^2}{aa}$, but $\dot{S} = cyy\dot{x}$, therefore for yy ſubſti-

tute its equal in the expreſſion $cyy\dot{x}$, then $\dot{S} =$ Fig. $\frac{cb^2x^2\dot{x}}{aa}$, whoſe Fluent is $S = \frac{cb^2x^3}{3aa}$, but $b = \frac{ay}{x}$, 37. and $bb = \frac{a^2y^2}{xx}$, now for bb ſubſtitute $\frac{a^2y^2}{xx}$ in the expreſſion above, then $S = \frac{ca^2y^2x^3}{3aaxx} = \frac{cyyx}{3}$, and when C$o$ flows to D, mn will coincide with AB, then $y = b$ and $x = a$, therefore $\frac{cbba}{3} = S$, the ſolidity of the whole cone ACB = area of the baſe multiplied by $\frac{1}{3}$ altitude. Wherefore the circumſcribing cylinder is to the inſcribed cone as 3 to 1.

EXAM. 4.

To find the ſolidity of a ſphere.

Let the diameter $AB = a$, $Ab = x$, $bc = y$, then 38. $bB = a - x$, by property of the figure $Ab \times bB = \overline{bc}^2$, that is, $ax - xx = yy$, but $\dot{S} = cyy\dot{x}$, for yy ſubſtitute $ax - xx$ its equal, then $\dot{S} = cax\dot{x} - cx^2\dot{x}$, whoſe Fluent is $S = \frac{cax^2}{2} - \frac{cx^3}{3}$, now if we conceive A$b$ in a flowing ſtate, and to flow or move till it arrives at B, then x becomes $= a$, and $S = \frac{ca^3}{2} - \frac{ca^3}{3} = \frac{3ca^3 - 2ca^3}{6} = \frac{ca^3}{6}$ the ſolidity of the whole ſphere. And the ſphere is to its circumſcribing cylinder as 2 to 3. And to the cone as 2 to 1.

Let $AB = 12 = a$, then $\frac{ca^3}{6} = 904.7808$ the ſolidity of a 12 inch globe.

EXAM. 5.

To find the ſolidity of a parabolic conoid generated by the revolution of the ſemi-parabola EnBF *about its axis* EF.

Put

Fig. Put EF $= b$, AB $= d$, Eo $= x$, $on = y$, then by 39, nature of the parabola $ax = yy$, but $\dot{S} = cyy\dot{x}$ the Fluxion of the variable ſolidity Eon, for yy put ax, then $\dot{S} = cax\dot{x}$, whoſe Fluent is $S = \frac{caxx}{2} = \frac{cyyx}{2}$; now if we conceive the ordinate on in a flowing ſtate, and to move till it arrives at, or coincides with the ordinate FB, then x becomes $= b$, and $y = d$, and $\frac{cyyx}{2}$ is $= \frac{cddb}{2}$ the ſolidity of the whole paraboloid, $=$ area of the baſe multiplied by half the height $=$ half of its circumſcribing cylinder.

Let EF $= 50 = b$, AB $= 30 = d$, then $\frac{cddb}{2}$ $= 70685$ the ſolidity.

EXAM. 6.

To find the ſolidity of a ſemi-cubical paraboloid.

In the ſemi-cubical parabola $ax^2 = y^3$, or $ax^m = y^{m+1}$, then $x = \frac{y^{\frac{m+1}{m}}}{a^{\frac{1}{m}}}$, in Fluxions is $\dot{x} = \frac{m+1}{ma^{\frac{1}{m}}} y^{\frac{1}{m}}\dot{y}$, but $\dot{S} = cyy\dot{x}$, put $b = \frac{m+1}{ma^{\frac{1}{m}}}$, then we ſhall have $\dot{S} = cyy\dot{x} = bcy^{\frac{5}{2}}\dot{y}$ the Fluxion of the ſolidity, whoſe Fluent is $= \frac{2}{7}bcy^{\frac{7}{2}} = \frac{2}{7}bcy^2\sqrt{y^3} = \frac{3}{7}cyyx$.

Let $x = 8$, and $y = 12$, then $\frac{3}{7}cyyx = 1551 =$ $\frac{3}{7}$ its circumſcribing cylinder.

It has been proved in *Ex.* 5, above, that a parabolic conoid is equal to half its circumſcribing cylinder, the ſame may be demonſtrated thus;

Fig.

Exam. 7.

Put EF $= x$, $a =$ *parameter*, $c = .7854$, *the reſt as above*, then the area that AB deſcribes by its revolution is $=4cyy$, this drawn into $\dot{x}$ is $4cyy\dot{x} = \dot{S}$, but by nature of the parabola $ax = yy$, in Fluxions is $a\dot{x} = 2y\dot{y}$, whence $\dot{x} = \frac{2y\dot{y}}{a}$, for $\dot{x}$ put $\frac{2y\dot{y}}{a}$ in the expreſſion $4cyy\dot{x}$, then $\dot{S} = \frac{8cy^3\dot{y}}{a}$, whoſe Fluent is $S = \frac{8cy^4}{4a}$, but $a = \frac{yy}{x}$, and $4a = \frac{4yy}{x}$, then for $4a$ in the expreſſion $\frac{8cy^4}{4a}$ ſubſtitute $\frac{4yy}{x}$, then $S = \frac{8cy^4}{\frac{4yy}{x}} = \frac{8cxy^4}{4yy} = 2cy^2x$, but it is evident the content of the cylinder is $= 4cy^2x$, therefore the parabolic conoid $2cy^2x$ is half of $4cy^2x$ its circumſcribing cylinder. 39.

Exam. 8.

To find the ſolidity of a ſpheroid formed by the revolution of the ſemi-ellipſis ACB *about its axis* AB. 40.

Let AD $= t$, CD $= a$, D$d = x$, $dc = y$, then AD $= t - x$, and B$d = t + x$.

Now by property of the ellipſis $\overline{dc}^2 : Ad \times dB :: \overline{CD}^2 : \overline{AD}^2$, that is, $yy : tt - xx :: aa : tt$, whence $ttyy = \overline{tt - xx} \times aa$, and $yy = \frac{t^2a^2 - a^2x^2}{tt} = aa - \frac{a^2x^2}{tt}$; but $\dot{S} = cyy\dot{x}$, then for yy ſubſtitute its equal, then $\dot{S} = caa\dot{x} - \frac{caax^2\dot{x}}{tt}$ the Fluxion of the ſolidity, Fig. 40. dity, whoſe Fluent is $S = caax - \frac{ca^2x^3}{3tt}$; now if we conceive dc in a flowing ſtate, and to move till it arrives at, and coincides with CD, then x becomes $= t$, and therefore $S = ca^2t - \frac{ca^2t^3}{3tt} = ca^2t - \frac{ca^2t}{3} = \frac{3ca^2t - ca^2t}{3} = \frac{2ca^2t}{3}$ the ſolidity of the ſemi-ſpheroid, equal to $\frac{2}{3}$ of its circumſcribing cylinder, and when $a =$ the whole conjugate diameter, and t the whole tranſverſe, then $\frac{ca^2t}{6} =$ ſolidity of the whole ſpheroid.

Let $AB = 60 = t$, $CG = 34 = a$, then $\frac{1}{6}ca^2t = 36316.896$ the ſolidity of the whole ſpheroid.

EXAM. 9.

To find the ſolidity of the fruſtum of a cone.

41. Put $a = EF = As$, $AB = b$, $CD = d$, $c = .7854$, $f = d - b$; $Eo = Ar = x$, then by the ſimilar triangles AsC, Arm, we have $a : f :: x : \frac{fx}{a} = 2mr$, then $b + \frac{fx}{a} = mn$; now $b + \frac{fx}{a}$ or $\frac{ab + fx}{a}$ ſquared is $= \frac{a^2b^2 + 2abfx + f^2x^2}{aa}$ which multiplied by c is $= \frac{ca^2b^2 + 2acbfx + cf^2x^2}{aa}$ and this drawn into $\dot{x}$ is $\dot{S} = \frac{ca^2b^2\dot{x} + 2acbfx\dot{x} + cf^2x^2\dot{x}}{aa}$, the Fluxion of the ſolidity, whoſe Fluent is $S = cb^2x + \frac{cbfx^2}{a} + \frac{cf^2x^3}{3aa}$; now if we conceive Eo in a flowing ſtate, and to move to F, then x becomes $= a$, and $S = cb^2a + cbfa + \frac{cf^2a}{3} = \overline{bb + bf + \frac{f^2}{3}} \times$

ca; but $f = d - b$, which put, then $S = b^2 + bd$ Fig. 41.
$- bb + \frac{d^2 - 2db + bb}{3} \times ca = 3bd + d^2 - 2db +$
$bb \times \frac{ca}{3} = \overline{dd + bb + bd} \times \frac{ca}{3}$, the ſolidity of the fruſtum.

Let $AB = 2 = b$, $CD = 4 = d$, $EF = 9 = a$, then $\overline{dd + bb + bd} \times \frac{ca}{3} = 65.9736$ the fruſtum's ſolidity.

EXAM. 10.

To find the ſolidity of a body bounded by four equal iſoſceles, triangular faces.

Let ABCD repreſent the ſolid of four equal and regular faces, each of which, namely, ACB, CAD, is an iſoſceles triangle. 42.

Let the baſe $AB = CD = a$, ſide $AC = AD = BC = b$, then the perpendicular of one of the faces, *viz.* $Cc = \sqrt{bb - \frac{aa}{4}}$, and the perpendicular of the ſolid is $\sqrt{bb - \frac{aa}{2}}$ which put $= c$, put $rm = x$, then $mn = a - x$ the other ſide of the generating parallelogram, and $\overline{a - x} \times x = ax - xx$ is the area of the generating plane; then $a : c :: x : \frac{cx}{a} =$ the variable perpendicular, whoſe Fluxion is $\frac{c\dot{x}}{a}$, and $\overline{ax - xx} \times \frac{c\dot{x}}{a} = \frac{cax\dot{x}}{a} - \frac{cx^2\dot{x}}{a} =$ $cx\dot{x} - \frac{cx^2\dot{x}}{a} = \dot{S}$, the Fluxion of the variable ſolidity, whoſe Fluent is $S = \frac{cx^2}{2} - \frac{cx^3}{3a}$, now if we conceive

Fig. 42. conceive *rm* in a flowing ſtate, and to move till it arrives at, and coincides with CD, then x will become $= a$, and we ſhall have $S = \frac{ca^2}{2} - \frac{ca^3}{3a} = \frac{ca^2}{2} - \frac{ca^2}{3} = \frac{ca^2}{6}$ for the ſolidity $= \frac{1}{6}$ of the circumſcribing priſm.

Let $AB = CD = 3 = a$, $AC = BC = 5 = b$; then the perpendicular of the ſolid is $c = 4.52769$, and $\frac{caa}{6} = 6.79153$ the ſolidity. See this done another way at page 83, of my Introduction to Fortification and Gunnery.

EXAM. II.

To find the ſolidity of a priſmoid whoſe baſes are parallel one to another, but diſproportional.

43. Let $DF = a$, $BH = b$, $CD = c$, $AB = d$; perpendicular height $DM = p$, $DN = x$, then by ſimilar triangles $DM : DN :: BH - DF : IO$, that is, $p : x :: b - a : \frac{b-a}{p}x = IO$; again, $DM : DN :: BC - AG : OL$, that is, $p : x :: d - c : \frac{d-c}{p}x = OL$, then $GO = a + \frac{b-a}{p}x$, and $OK = c + \frac{d-c}{p}x$, but $\overline{a + \frac{b-a}{p}x} \times \overline{c + \frac{d-c}{p}x} \times \dot{x}$ is the Fluxion of the ſolidity, *viz.* $\dot{S} = ac\dot{x} + \frac{d-c}{p}ax\dot{x} + \frac{b-a}{p}cx\dot{x} + \overline{b-a} \times \frac{d-c}{pp}x^2\dot{x}$, whoſe Fluent is $S = acx + \frac{d-c}{2p}ax^2 + \frac{b-a}{2p}cx^2 + \frac{\overline{b-a} \times \overline{d-c}}{3pp}x^3$ for the ſolidity of KOGE; now if

we conceive DN to be in a flowing ſtate, and to flow to DM, then x becomes $= p$, and the ſolidity of the whole priſmoid BE is DF × AB + BH × CD + 2BH × AB + 2DF × CD $: \times \frac{p}{6}$, that is, Fig. 43.

$$ad + bc + 2bd + 2ac : \times \frac{p}{6} = bd + \tfrac{1}{2}ad + ac$$

$$+ \tfrac{1}{2}bc : \times \frac{p}{3} = \overline{b+\tfrac{1}{2}a} \times d + \overline{a+\tfrac{1}{2}b} \times c : \times \frac{p}{3}.$$

Let AB $= 38 = d$, DF $= 12 = a$, CD $= 30 = c$, BH $= 16 = b$, DM $= 72 = p$, then $\overline{b+\tfrac{1}{2}a} \times d + \overline{a+\tfrac{1}{2}b} \times c : \frac{p}{3} = 34464$ the ſolidity of the whole priſmoid.

EXAM. 12.

To find the ſolidity of a parabolic ſpindle AmCnBD, *revolving round the ordinate* CD.

Let axis DB $= b$, ordinate CD $= m$, pB $= x$, E$n = y = b - x$, $np = z$, then by property of the curve $ax = zz$, and $x = \frac{zz}{a}$; now $y = b - x$ in Fluxions is $\dot{y} = -\dot{x}$, and $x = \frac{zz}{a}$ in Fluxions is $\dot{x} = \frac{2z\dot{z}}{a}$, then $\dot{y} = -\dot{x} = -\frac{2z\dot{z}}{a}$, but $\dot{S} = 2czy\dot{y}$, for $\dot{y}$ put $-\frac{2z\dot{z}}{a}$, then $2czy\dot{y}$ becomes $\frac{-4cz^2y\dot{z}}{a}$, but the Fluent of $\dot{y} = -\dot{x} = -\frac{2z\dot{z}}{a}$ is $y = -x = -\frac{zz}{a}$, which put for y in the ex- 44.

preſſion

Fig. 44. preſſion $\frac{-4cz^2y\dot{z}}{a}$, then it becomes $\frac{4cz^2\dot{z}}{a}\times\overline{b-\frac{z^2}{a}}$ $=\frac{-4cbz^2\dot{z}}{a}+\frac{4cz^4\dot{z}}{aa}=\dot{S}$, whoſe Fluent is $\frac{-4cbz^3}{3a}$ $+\frac{4cz^5}{5aa}$, but in C, where S $=$ o, $z=m$, therefore by Prop. XXIV. the Fluent corrected is $S=\frac{4cbm^3}{3a}$ $-\frac{4cbz^3}{3a}+\frac{4cz^5}{5aa}-\frac{4cm^5}{5aa}=\frac{8cbm^3}{15a}-\frac{4cz^3}{15a}\times\overline{5b-3x}$ for the ſolid C*nprm*, but when $x=0$, and $z=0$, the ſolidity of the whole parabolic ſpindle is $\frac{8cbm^3}{15a}=\frac{8cmbb}{15}=\frac{8}{15}$ area of the baſe multiplied by the length or height.

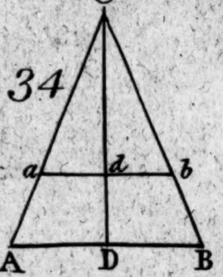
Let CD $=108=m$, DB $=18=b$, then $\frac{8cbbm}{15}$ $=58629.7958$ the ſolidity.

EXAM. 13.

To find the ſolidity of a hyperbolic conoid formed by the revolution of the hyperbola AMCB *about its axis* AB.

45. Let AB $=a$, BC $=d$, tranſverſe axis AN $=2b$, then NB $=a+2b$; alſo put AP $=x$, PM $=y$, then NP $=2b+x$, and by the property of the hyperbola NB $\times$ AB : $\overline{BC}^2$: : NP $\times$ AP : $\overline{PM}^2$, that is, $aa+2ab:dd::xx+2bx:\frac{ddxx+2bddx}{aa+2ab}=yy$, but $\dot{S}=cyy\dot{x}$ the Fluxion of the variable ſolid AMP, now for yy ſubſtitute its value above, then $\dot{S}=\frac{cddx^2\dot{x}+2cd^2bx\dot{x}}{aa+2ab}$ whoſe Fluent is $S=\frac{cd^2x^3}{3aa+6ab}$ $+\frac{2bcddx^2}{2aa+4ab}$; now if we conceive the ordinate PM in a flow-

34



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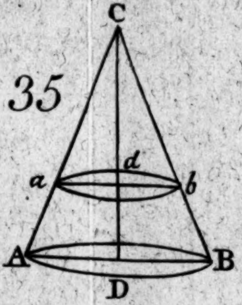

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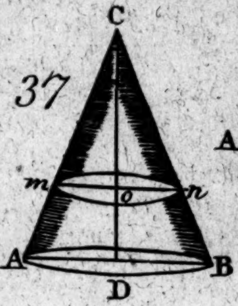

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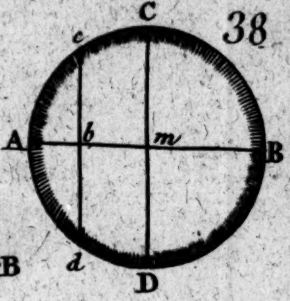

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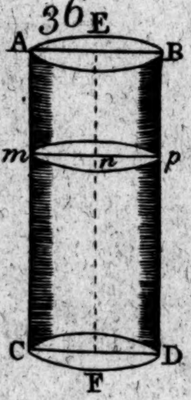

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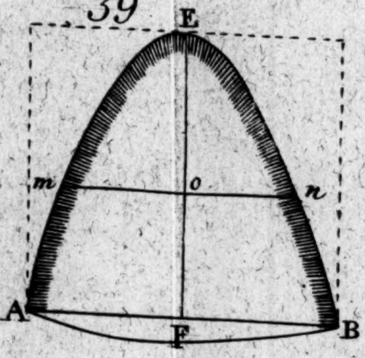

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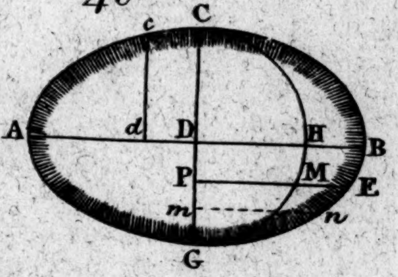

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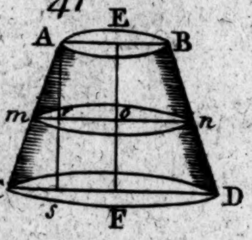

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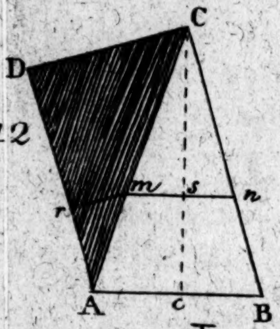

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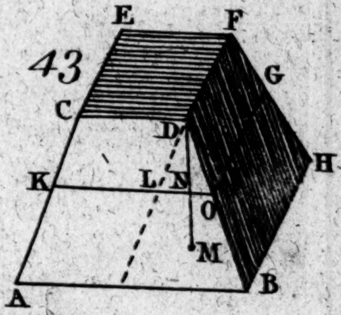

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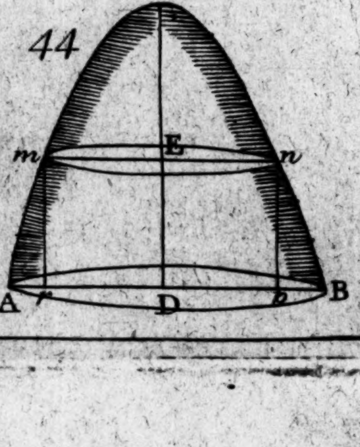

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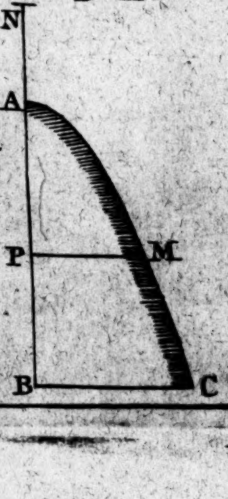

Fig.

a flowing ſtate, and to flow till it arrives at, or coincides with the ordinate BC, then AP will become $= AB$ or $x = a$, and we ſhall have $\frac{cdda^2}{3a+6b} + \frac{cddba}{a+2b}$, the ſolidity of the whole hyperboloid.

Let $AB = 10 = a$; $BC = 12 = d$, $NA = 30 = 2b$, then $\frac{cdda^2}{3a+6b} + \frac{cddba}{a+2b} = 2073.456$ the ſolity of the hyperboloid.

Exam. 14.

Let the ſemi-ellipſis CBG, *and the ſemi-circle* CHG, *both be conceived to revolve about the axis* CG; *the ſphere generated by the revolution of the ſemi-circle, will be to the ſpheroid deſcribed by the revolution of the ſemi ellipſis as* $\overline{DH}^2$ *to* $\overline{DB}^2$.

40. Draw PE parallel to BD, and *mn* infinitely near to PE; let $GP = x$, $PM = y$, $DG = DH = r$; $DB = t$, then $CP = 2r - x$; and let $\frac{r}{p}$ expreſs the ratio of the radius to the periphery, then $\frac{py}{r}$ will be the periphery of a circle deſcribed with the radius PM; moreover from the nature of the ellipſis, $r : t :: y : \frac{ty}{r} = PE$, hence the periphery of the circle deſcribed with the radius $PE = \frac{pty}{rr}$, whoſe area is $\frac{pt^2y^2}{2r^3}$, which multiplied by $\dot{x}$ (G*m*) gives $\frac{pt^2y^2\dot{x}}{2r^3}$ the Fluxion of the ſpheroid; alſo the area of the circle deſcribed with the radius PM is

Fig. $= \frac{pyy}{2r}$, which drawn into $\dot{x}$ gives $\frac{pyy\dot{x}}{2r}$ the Fluxion of the ſphere; but CP × PG $= \overline{PM}^2$, that is, $yy = 2rx - xx$, which ſubſtitute for yy in both the above expreſſions, then $\frac{pt^2y^2\dot{x}}{2r^3}$ becomes $\frac{prt^2x\dot{x}}{r^3} - \frac{pt^2x^2\dot{x}}{2r^3}$, and $\frac{py^2\dot{x}}{2r}$ becomes $\frac{2prx\dot{x} - px^2\dot{x}}{2r}$ the Fluent of the former is $\frac{prt^2x^2}{2r^3} - \frac{pt^2x^3}{6r^3}$, and of the latter is $\frac{prx^2}{2r} - \frac{px^3}{6r}$, now when $x = 2r$, then the whole ſpheroid will be to the whole ſphere, as $\frac{2pt^2r^3}{r^3} - \frac{8pt^2}{6r^3}$ to $\frac{2pr^3}{r} - \frac{8pr^3}{6r}$, or as $4t^2$ to $4r^2$, that is, as t^2 to r^2, or in the duplicate ratio as $\overline{DB}^2$ to $\overline{DH}^2$.

And by the ſame way of reaſoning it is evident, that a ſphere, deſcribed with the radius of the greater ſemi-axis of the ellipſis, is to the ellipſoid, in the duplicate ratio of the greater ſemi-axis to the leſſer.

I ſhall here inſert ſome eaſy promiſcuous Problems, with their Solutions, for the benefit of my readers, which will further evince the uſe of Fluxions.

PROB. I.

There is a ſtone whoſe length, with twice the breadth and three times the depth, is 144 inches, and its ſolidity is $10\frac{2}{3}$ feet; required the ſtone's dimenſions.

Put $144 = a$, $10\frac{2}{3}$ feet $= 18432$ inches $= b$, $x =$ length, $y =$ breadth, and $z =$ depth; then $\frac{a - x - 2y}{3} = z$, and $\frac{b}{xy} = z$, whence $\frac{a - x - 2y}{3}$

$=$

$= \frac{b}{xy}$, and $axy - xxy - 2xyy = 3b$; here are two unknown quantities and but one equation; now conceive x in a flowing ſtate, then the above equation in Fluxions is $ay\dot{x} - 2yx\dot{x} - 2y^2\dot{x} = 0$; divide by $y\dot{x}$, then $a - 2x = 2y$, which multiply by 2 is $2a - 4x = 4y$. Again, let y flow, then $axy - xxy - 2xyy = 3b$, in Fluxions is $ax\dot{y} - x^2\dot{y} - 4xy\dot{y} = 0$, divide by $x\dot{y}$ is $a - x = 4y$, hence $2a - 4x = a - x$, and $x = \frac{a}{3} = 48$ inches the length, $y = \frac{a}{2} - x = 24$ the breadth, and $z = \frac{b}{xy} = 16$ the depth.

PROB. II.

Required that curve whoſe ſubnormal is $= a$, *an invariable line.*

Put abſciſſa AP $= x$, PM $= y$, then by *Prob.* II. 22.
we have $\dot{x} : \dot{y} :: y : \frac{y\dot{y}}{\dot{x}}$ = PB the ſubnormal $= a$, by the queſtion, then $y\dot{y} = a\dot{x}$, whoſe Fluent is $\frac{yy}{2} = ax$, or $2ax = yy$ the equation to the curve, which is the common parabola, and half the parameter is a.

PROB. III.

To find the nature of the curve whoſe ſubtangent is $\frac{\overline{m+n} \times \overline{ax+xx}}{ma \pm \overline{m+n} \times x}$.

By *Prob.* II. we have $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}}$ the ſub-

Fig. tangent $= \frac{dax \pm xx}{ma \pm dx}$, where $d = m + n$, hence $\frac{\dot y}{y} = \frac{ma\dot x \pm dx\dot x}{dax \pm xx}$, whoſe Fluent is log. $y = \pm d - \frac{m}{d}$ $: \times$ log. $da \pm x$, which is, being properly reduced, $xy^d = \overline{da \pm x}|^{\pm dd - m}$ the equation of the curve.

PROB. IV.

To inſcribe the greateſt parallelogram in the common parabola ADF.

46. Put $a =$ parameter, AC $= b$, AB $= x$, then BC $= b - x$, and $ax = yy$ by nature of the parabola, where $x = \frac{yy}{a}$; but BC × BD = area of the rectangle BDCE, that is, $\overline{b - x} \times y = by - yx$, now for x ſubſtitute $\frac{yy}{a}$ its value, then $by - \frac{y^3}{a}$ will be the area a *maximum*, in Fluxions is $b\dot y - 3y^2\dot y = 0$, a *max*. dividing by $\dot y$ is $ab = 3y^2$, whence $y = \sqrt{\frac{ab}{3}}$, and $x = \frac{b}{3}$.

PROB. V.

To inſcribe the greateſt parallelogram in a quadrant of a circle.

47. Put radius AC $=$ CD $= r$, BC $= x$, then BD $= \sqrt{rr - xx}$, but BD × BC $= x\sqrt{rr - xx}$, the area of the rectangle BDEC, which is to be a *maximum*, in Fluxions is $\frac{rr\dot x - 2x^2\dot x}{\sqrt{rr - xx}} = 0$, whence $rr\dot x - 2x^2\dot x = 0$, divide by $\dot x$ is $rr - 2xx$ and $x = r\sqrt{\tfrac{1}{2}}$, which being known, BD $= \sqrt{rr - xx}$ will be known.

PROB.

PROB. VI.

To inſcribe the greateſt rectangle in an equilateral hyperbola.

Let ADF be an equilateral hyperbola whoſe ſemi-axis is GA, put GA $= a$, GC $= b$, GB $= x$, then by property of the hyperbola, we have $x : a :: a : \frac{aa}{x} =$ GT, whence BC $= b - x$, and by nature of the hyperbola, the ſquare of any perpendicular BD is to the rectangle of the correſponding parts of that axis (ſee *Emerſon*'s Conic Sections) as the ſquare of half the ſecond axis is to the ſquare of half the firſt, that is, BD $= \sqrt{xx-aa}$, whence the area of the rectangle BE will be $\overline{b-x} \times \sqrt{xx-aa}$, a *maximum*, in Fluxions is 48.

$$\frac{4x^3\dot{x}-6bx^2\dot{x}-2b^2x\dot{x}-2a^2x\dot{x}+2a^2b\dot{x}}{2\overline{b-x}\sqrt{xx-aa}} = 0,$$ or $4x^3\dot{x} - 6bx^2\dot{x} - 2b^2x\dot{x} - 2a^2x\dot{x} + 2a^2b\dot{x} = 0$, dividing by $2x\dot{x} - 2b\dot{x}$ is $2xx - bx - aa = 0$, whence by compleating the ſquare and extracting the root, we get $x = \frac{1}{4}b + \sqrt{\frac{aa}{2} + \frac{bb}{16}}$, then BC $= b - x$ is known, and conſequently BD $= \sqrt{xx-aa}$.

PROB. VII.

Required that curve whoſe indefinite quadrature is expreſſed by $\frac{x^3}{a}$.

The Fluxion of $\frac{x^3}{a}$ is $\frac{3x^2\dot{x}}{a}$, and $y\dot{x}$ is the Fluxion of the area of any curve by *Prob.* II. therefore $y\dot{x} = \frac{3x^2\dot{x}}{a}$ by the queſtion, or $ay\dot{x} = 3x^2\dot{x}$, dividing

Fig. viding by $3\dot{x}$ we have $x^2 = \frac{ay}{3}$, therefore the curve ſought is the exterior part of the *Apollonian* parabola whoſe parameter is $\frac{1}{3}a$.

49. For $CA = DB = x$, $AB = CD = y$, then by property of the parabola $\frac{1}{3}ay = xx$, now it is plain, that the area of the parabola C*m*BD is $\frac{2}{3}xy$, therefore C*m*BAC is $\frac{1}{3}xy$, but $y = \frac{3x^2}{a}$, which ſubſtitute for y in the above expreſſion, gives $\frac{1}{3}x \times \frac{3x^2}{a} = \frac{x^3}{a}$ the area of CAB*m*C.

PROB. VIII.

Let $x^3 - y^4 = ax^2y$ *expreſs the relation of* CD *to* DB. *Now there is given the curvilineal area* $CDB = 556\frac{11}{70}$, *together with the invariable quantity* $a = 3$, *to find the abſciſſa* CD *and ordinate* DB.

49. Put $556\frac{11}{70} = b$, the equation of the curve is $x^3 - y^4 = ax^2y$, for x write zy, then $x^3 = z^3y^3$, which ſubſtitute for x^3, and the equation becomes $z^3y^3 - y^4 = az^2y^3$, divide by y^3 is $z^3 - y = az^2$, therefore $y = z^3 - az^2$, and zy or $x = z^4 - az^3$, which in Fluxions is $\dot{x} = 4z^3\dot{z} - 3az^2\dot{z}$, then $y\dot{x} = \overline{4z^3\dot{z} - 3az^2\dot{z}} \times \overline{z^3 - az^2} = 4z^6\dot{z} - 7az^5\dot{z} + 3a^2z^4\dot{z}$ the Fluxion of the area, whoſe Fluent is $\frac{4z^7}{7} - \frac{7az^6}{6} + \frac{3a^2z^5}{5} = b$, or $120z^7 - 735z^6 + 1134z^5 = 116793$, whence $z = 4$, $x = 64$, the abſciſſa, and $y = 16$ the ordinate.

PROB.

PROB. IX.

Admit Stamford *bears due ſouth from* Doncaſter 70 *miles, and* Sheffield *due weſt from* Doncaſter, *ſuppoſe* 30 *miles*; *a meſſenger is diſpatched from* Sheffield *to* Stamford *in order to perform his journey in the ſhorteſt time poſſible, but by reaſon of bad roads he can travel no more than* 3 *miles an hour until he arrives at* Tuxford, *ſituated in a direct line between* Doncaſter *and* Stamford, *and then he can travel* 5 *miles an hour*; *required the time of his performing the whole journey, and the diſtance between* Doncaſter *and* Tuxford.

Let A denote *Sheffield*; D, *Doncaſter*; T, *Tuxford*; and S, *Stamford*. Then there is given DS $= 70 = m$, AD $= 30 = a$, $p = 3$, $q = 5$, the miles travelled *per* hour, and put $x =$ DT, then TS $= m - x$, and AT $= \sqrt{aa+xx}$, then will the time of deſcribing AT $= \frac{1}{p}\sqrt{aa+xx}$, that of TS $= \frac{m-x}{q}$, 50.

and the whole time $= \frac{1}{p}\sqrt{aa+xx} + \frac{m-x}{q}$, a *minimum*, whoſe Fluxion is $\frac{x\dot{x}}{p\sqrt{aa+xx}} - \frac{\dot{x}}{q} = 0$, divide by $\dot{x}$ gives $\frac{x}{p\sqrt{aa+xx}} = \frac{1}{q}$, whence $qx = p\sqrt{aa+xx}$, and $x = \frac{ap}{\sqrt{qq-pp}} = \frac{3}{4}a = 22\frac{1}{2}$ miles, the diſtance between *Doncaſter* and *Tuxford*, hence AT $= \frac{5a}{4} = 37\frac{1}{2}$ from *Sheffield* to *Tuxford*, time of going it $12\frac{1}{2}$ hours; and from *Tuxford* to *Stamford*, $\frac{m-x}{q} = 9\frac{1}{2}$ hours, and the diſtance between *Tuxford* and *Stamford* is $47\frac{1}{2}$ miles; and therefore he may finiſh his journey in 22 hours.

PROB. X.

Required the area of the greateſt parallelogram inſcribed in the ſegment of a circle, whoſe chord is 40, *and verſed ſine is* 38.

51. Given AF = 20, EF = 38, hence $\frac{\overline{AF}|^2}{2EF} + \frac{EF}{2}$ = ID = 24.263157 = n, and EF — EI = IF = 13.736843 = m.

Let KD = x, IK = y, then $n^2 = x^2 + y^2$, which in Fluxions is $x\dot{x} + y\dot{y} = 0$, therefore $\dot{y} = \frac{-x\dot{x}}{y}$, but the *maximum* is $\overline{m+y} \times x = mx + xy$, therefore its Fluxion $y\dot{x} + x\dot{y} + m\dot{x} = 0$, in which for $\dot{y}$ and $x\dot{x}$ write their values, and we have $2y^2 + my = n^2$, therefore $y = \frac{\sqrt{8n^2+m^2}-m}{4} = 14.062766$, hence $x = 19.772187$, and the area BGHD = 1099.318135 a *maximum*.

PROB. XI.

There is a parabolical ſolid of good oak, formed by the rotation of the curve of a parabola, about the axis at the vertex, which therefore is the axis of the body. If this ſolid be placed horizontally with the thick end in a wall, it is required to find the latus rectum *of the generating* parabola, *ſo that ſuch a ſolid* 20 *feet in length ſhall juſt break by its own weight.*

52. Let AB = x, AC or AD = y, parameter = a, then $ay = xx$; $c = 3.1416$, in this ſolid the content is $\frac{cxyy}{5}$, the diſtance of the center of gravity G, from the baſe A = $\frac{1}{6}x$.

By *Emerſon's* Mechanics.

1. The ſtrength of a cylinder is about $\frac{47}{64}$, the

ſtrength of a parallelopipedon whoſe ſide is equal the diameter, (being a very little leſs than the area of the ſection, which is $\frac{.7854}{1}$).

2. A cubic inch of oak weighs .463 ounces = ,029lb.

3. 165lb. ſuſpended at the end of a piece of oak an inch ſquare and $1\frac{1}{2}$ foot long, will juſt break it.

Hence the ſtrength at A will be $\frac{47}{64} \times 8y^3 = \frac{47}{8}y^3$, and if the ſolid DBC be ſuſpended at G, the ſtreſs at A will be $\frac{1}{6}x \times \frac{1}{5}cxyy \times .029$lb. and therefore by the principles of Mechanics, the ſtrength will be

(ſtrength) inch (ſtreſs) (ſtrength)

as the ſtreſs, that is, $1^3 : 18 \times 165 :: \frac{47}{8}y^3 : \frac{1}{6}x \times \frac{1}{5}cxyy \times .029$; that is, $1 : 2970 :: \frac{47}{8}y : \frac{,0911xx}{30} :: \frac{47}{8}y :: \frac{.0911av}{30}$; that is, $1 : 2970 :: 47 \times 30 : 8 \times .0911a$.

Therefore $a = \frac{2970 \times 47 \times 30}{8 \times .0911} = 5746020$ inches, and $y = \frac{xx}{a} = .01$, and $2y = \frac{1}{50}$ inch, the thickneſs of the ſolid at A, that it may break by its own weight.

Cor. Since neither x nor y is concerned in the equation, $a = \frac{2970 \times 47 \times 30}{8 \times .0911}$; therefore the parameter is the ſame for any length x, as well as for 20 feet; that is, the ſame parabolic ſolid will equally break at any other length as well as that of 20 feet; and the parabolic conoid will conſequently be equally ſtrong every way, or throughout to bear its own weight.

PROB.

PROB. XII.

There is a meadow bounded by three equal and ſimilar curve lines, whoſe convexities are all turned towards one another, the length of each curve line is given = 50 7786 chains per Gunter, and the equation expreſſing the nature of each curve is $y^2 = \frac{3}{214}x^3$, *the meadow's content is required.*

53. Let $m = 50.7786$ the length of the curve, AB $= x$, and BC $= y$, then ſince $y^2 = \frac{3}{214}x^3$, or $y^2 = ax^3$, we get $y = x\sqrt{ax}$, and $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}}{2}\sqrt{4 + 9ax}$, whoſe Fluent by *form* 3 is $\frac{\overline{4 + 9ax}|^{\frac{3}{2}} - 8}{27a}$ $= m$, the length of the curve AC, hence $x = \frac{\overline{27ma + 8}|^{\frac{2}{3}} - 4}{9a}$ = AB = 40.016462, y = BC = 29 971732 and AC = 49.996229 $= 2n$.

Again, $y = x\sqrt{ax}$, which multiplied by $\dot{x}$ is $y\dot{x} = x\dot{x}\sqrt{ax}$ the Fluxion of the area, whoſe Fluent by *form* 3 is $\frac{2xx\sqrt{ax}}{5}$, and by ſubſtituting y for $x\sqrt{ax}$ is $\frac{2}{5}xy$ the area of the curve ABCEA, hence the area of ACE $= \frac{1}{10}xy$, and the area of the meadow $= nn\sqrt{3} - \frac{3}{10}xy = 722.55936$ chains ſquare

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= 72 1 0.95.

PROB. XIII.

A flexible chain eighteen inches long was hung horizontally on two pins juſt 12 inches aſunder; required a theorem to find the loweſt deſcent of the chain, and the area of all ſuch curves.

Let the chain be ſuſpended on the two fixt points A and B, and diſpoſe itſelf into the curve ACB, C its vertex, CD its axis, and EF an ordinate; draw the tangent FG, and FH $=$ FC perpendicular to the horizon and GH parallel to it; then let the tenſion in C be repreſented by GH $= a$, CE $= x$, EF $= y$, CF $= z$; make F*n* $=$ F*f*, draw F*m* perpendicular, and *mn* parallel to FG, then $af = mn = \dot{x}$, $a\mathrm{F} = \dot{y}$, ſee page 151 *Emerſon's* Fluxions, whence by ſimilar triangles $z : a :: \dot{x} : \dot{y}$, therefore $a\dot{x} = z\dot{y}$, which is one property of this curve called the *catenaria*; ſince FH is the perpendicular force or weight of the line at F, and x the Fluxion of the tenſion of the line in F, and therefore the Fluent $x =$ tenſion or force acting in direction *mn*, but in C where $x = 0$, this tenſion $= a$, therefore by correction, the whole tenſion drawing in direction of the curve is $a + x$, and this is the force in F $=$ FG; therefore again by ſimilar triangles, $a + x : z :: \dot{z} : \dot{x}$, whence $a\dot{x} + x\dot{x} = z\dot{z}$, whoſe Fluent is $x^2 + 2ax = zz$, which is another property of this curve; therefore $a = \frac{zz - xx}{2x}$ the parameter, and $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{z}}{\sqrt{a^2 + z^2}}$, whoſe Fluent by *form* 9, is $2.30258a \times \log. \frac{z + \sqrt{aa + zz}}{a}$, and ſubſtituting the value of a, we have $y = \frac{z^2 - x^2}{2x}$ $\times$ 2.302

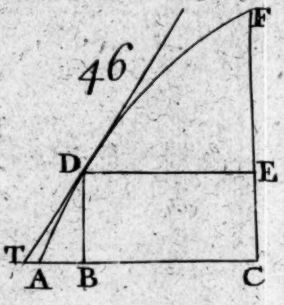
Fig. $\times\, 2.30258 \times \log. \frac{z+x}{z-x}$; now if we ſuppoſe E in a flowing ſtate, and to flow to D, then $x = \text{DC} =$ 6.0313901 the loweſt deſcent.

Again, $x\dot{y} = a\dot{z} - \frac{aa\dot{z}}{\sqrt{aa+zz}}$, whoſe Fluent by *form* 9 is $az - 2.30258aa \times \log. \frac{z+\sqrt{aa+zz}}{a}$, hence the area of the curve ſpace ACB $= 2xy + 2ay - 2az = \frac{\overline{zz+xx} \times \overline{y+x} - 2z^3}{x} = 50.1816$.

Another way.

55. Let A*b*D $= 9 = c = \text{AR}$, BD $= b = 6$, AB $= x$; drawing AR parallel to CD, and making AR $=$ AD, the half length of the curve, by property of the curve, the parameter AP $= \frac{cc-xx}{2x}$, ſee above; then by nature of the curve, we have $2.30258 \times \frac{c^2-x^2}{2x} \times \log. \frac{c+\sqrt{\frac{c^4-2c^2x^2+x^4}{4xx}+cc}}{\frac{c^2-x^2}{2x}} = b$, this being reduced by diviſion and extraction, will give $\frac{2bx}{2.30258 \times \log. c^2-x^2} = \log. \frac{c+x}{c-x}$, or $\frac{2bx}{c^2-x^2} = 2.30258 \times \log. \frac{c+x}{c-x} + \frac{2bx}{c^2-x^2} =$(Fluent of $\frac{2c\dot{x}}{c^2-x^2}$ $=) \frac{2x}{c} + \frac{2x^3}{3c^3} + \frac{2x^5}{5c^5} + \frac{2x^7}{7c^7}$, &c. that is $\frac{4x^2}{3c} + \frac{4x^4}{15c^3} + \frac{4x^6}{35c^5}$, &c. $= 6$, or by ſubſtitution $ax^2 + dx^4 + mx^6$, &c. $= p$, whence by *Theorem* 6 in reverſion of ſeries, $x = 6.03139$, and conſequently

AP $=$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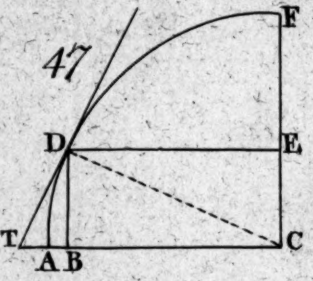
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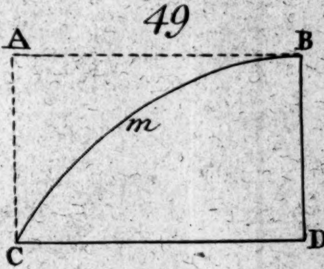
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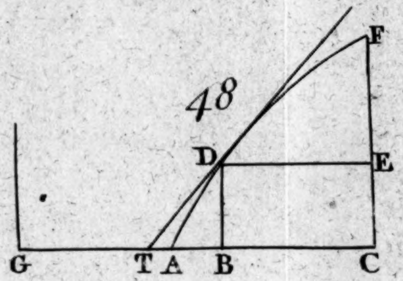
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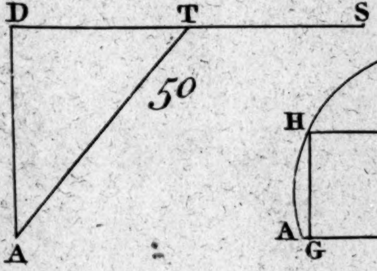
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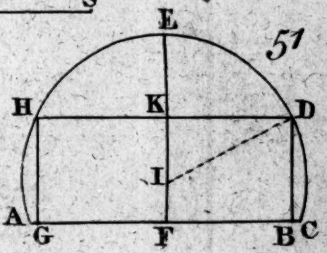
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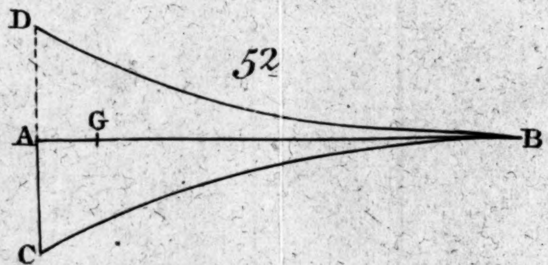
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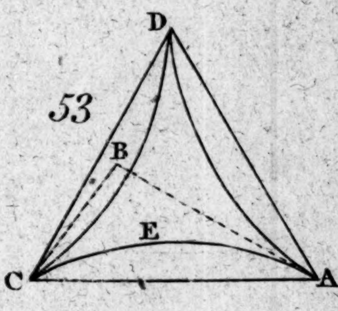
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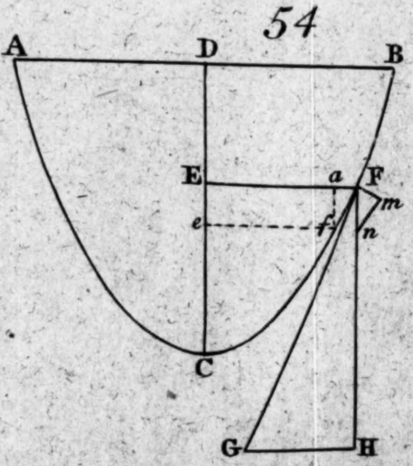
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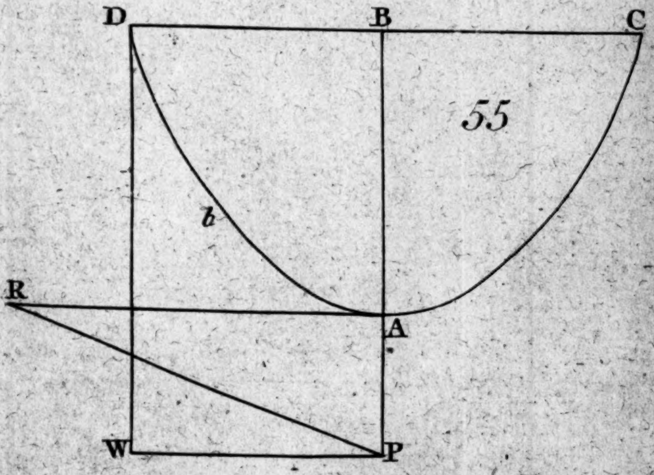
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$AP = 3.7$; but by *Huygen*'s Theorem, the area Fig.
WD*b*APW is equal to twice the area of the rectangular triangle PAR, which taken from the area of the rectangle PWDB, leaves the area of the ſemi-catenaria A*b*DBA $= 25.092$, conſequently the area of the catenaria $= 50.184$.

PROB. XIV.

To determine the law of the weights, which preſs each particle of a perfect flexible line in ſuch manner as that it ſhall form a curve, whoſe equation is $ax = y^4$.

Let a = force or tenſion of the line in the point of its loweſt deſcent, any abſciſſa $= x$, its ſemi-ordinate $= y$, and the arch correſpondent thereto $= z$; then $z\dot{y} = a\dot{x}$ (as is inveſtigated in the laſt Problem.) But by the *data* $ax = y^4$ whoſe Fluxion is $a\dot{x} = 4y^3\dot{y} = \dot{y}z$; therefore $z = 4y^3$ = the gravity of the line, or the required law of the weights preſſing perpendicularly every particle of the ſaid line, which is as *the cube of the ordinate.*

PROB. XV.

Let an arc of 30 *degrees of a circle whoſe diameter is* 100, *revolve round its ſine, required the curve's ſuperficies of the generated ſolid.*

Let $DC = a = 25$, $c = 3.1416$, then $AC = 2a$, 56.
$AD = a\sqrt{3} = 43.3 = n$, alſo put $x = CF$, $y = FG$, then $\overline{n+y}|^2 + \overline{a-x}|^2 = 4a^2$, therefore $x = a - \sqrt{a^2 - 2ny - y^2}$, and the Fluxion of the curve ſurface is $\frac{4cay\dot{y}}{\sqrt{a^2-2ny-y^2}}$; let $v^2 = a^2 - 2ny - y^2$,

and it is $\frac{4can\dot{v}}{\sqrt{4a^2 - v^2}} - 4ca\dot{v}$, whoſe Fluent, by *form* 10, is $4ca \times \overline{An - v}$, where $A = .017453 \times$ degrees in the arch, whoſe ſine is $\frac{v}{2a}$, and rad. $= 1$; which corrected is $4ca \times \overline{An - Bn + a - v}$, where B is the arch of 30 degrees, radius $= 1$; and when $y = 2a - n$, it is $2c : \times \overline{2a^2 - n} \times$ arch $BC = 731.20816$ the ſuperficies required.

PROB. XVI.

Given the perpendicular height of an Apothecary's mortar $= 9$ *inches, bottom diameter* $= 6$, *top diameter* $= 12$, *and the curvature of the* mortar's *ſides are ſuppoſed to be the* Apollonian parabola, *whoſe vertex is a point on the uppermoſt edge of the* mortar, *or extremity of the top diameter; required the content in wine gallons.*

57. Let AEHK repreſent the mortar generated by the revolution of the parabolic curve EK, about its axis BI parallel to DG; put $BE = 6 = a$, $BI = 9 = b$, $IK = 3 = n$, $c = 3.1416$, $x =$ any abſciſſa DG, then $EC = a - n = m$, and by Conics $BI : \overline{EC}|^2 :: DG : \overline{DE}|^2$, that is, $b : m^2 :: x : \frac{m^2 x}{b} = \overline{DE}^2$, therefore $FG = a - m\sqrt{\frac{x}{b}}$, and the Fluxion of the ſolid $= c\dot{x} \times \overline{a - m\sqrt{\frac{x}{b}}}|^2$, whoſe Fluent is $c \times : a^2 x + \frac{m^2 x^2}{2b} - \frac{4ma}{3}\sqrt{\frac{x^3}{b}}$, and when x flows $= b$, it gives the ſolidity of the mortar $= cb \times a^2$

$\times \overline{a^2 + \frac{1}{2}m^2 - \frac{4am}{3}} = \frac{cb}{6} \times \overline{a^2 + 3n^2 + 2ac} =$ Fig. 57.
466.5276 cubic inches $=$ 2.0196 wine gallons.

Another way.

Put $2b = AE = 12$, $2b = HK = 6$, $a = BI = 9$, $c = 3.1416$, $S =$ variable ſolidity HLGKI, $FI = x$, $FG = y$, then by property of the curve $a : \overline{b - b}|^2$ $(= n^2) :: a - x : \overline{b - y}|^2$, which gives $x = a - \frac{a}{n^2}$ $\times \overline{b - y}|^2$, which in Fluxions is $\dot{x} = \frac{2a}{n^2} \times \overline{b\dot{y} - y\dot{y}}$, this drawn into cyy the area of the greateſt plane, is $\frac{2ac}{nn} \times \overline{by^2\dot{y} - y^3\dot{y}} = \dot{S}$, whoſe Fluent is $\frac{2ac}{n^2} \times \frac{by^3}{3}$ $- \frac{y^4}{4} = S$, which when y flows $= b$ is $\frac{{}^{*}\,2ac}{n^2} \times :$ $\frac{b^4}{12} - \frac{bb^3}{3} + \frac{b^4}{4} = 466.5276$ cubic inches $=$ 2.0196 wine gallons.

* Solidity $= \frac{2ac}{n^2} \times : \frac{by^3}{3} - \frac{y^4}{4}$, but when $S = o$ $(x = o)$ and $y = IK = b$, and ſolidity $o = \frac{2ac}{n^2} \times : \frac{bb^3}{3} - \frac{b^4}{4}$ ſubtract; and the correct ſolid. $= \frac{2ac}{n^2} \times : \frac{by^3}{3} - \frac{y^4}{4} - \frac{bb^3}{3} + \frac{b^4}{4}$, and when $y = b$, total ſolidity $= \frac{2ac}{n^2} \times : \frac{b^4}{3} - \frac{b^4}{4} - \frac{bb^3}{3} + \frac{b^4}{4}$ which reduced is $\frac{2ac}{n^2} \times : \frac{b^4}{12} - \frac{bb^3}{3} + \frac{b^4}{4}$, the correct Fluent, as above.

PROB. XVII.

To rectify the common parabola, whoſe equation is $2px = yy$.

Firſt $x = \frac{y^2}{2p}$, in Fluxions $\dot{x} = \frac{y\dot{y}}{p}$, ſquared is $\dot{x}^2 = \frac{y^2\dot{y}^2}{p^2}$, and conſequently $\dot{z} = (\sqrt{\dot{x}^2 + \dot{y}^2} =)$ $\sqrt{\dot{y}^2 + \frac{y^2\dot{y}^2}{p^2}} = \frac{\dot{y}}{p}\sqrt{p^2+y^2} = \frac{\dot{y}}{p\sqrt{p^2+y^2}} \times p^2+y^2$, or rather $\dot{z} = \frac{p\dot{y}}{\sqrt{p^2+y^2}} + \frac{y^2\dot{y}}{p\sqrt{pp+yy}}$; now the Fluent of $\frac{p\dot{y}}{\sqrt{pp+yy}}$ is $= 2.30258 \text{ log. } y + \sqrt{p^2+y^2}$, by form 9, which call F, and by *Prop.* 21, that of $\frac{y^2\dot{y}}{p\sqrt{p^2+y^2}}$ is $= \frac{1}{2}y\sqrt{\frac{pp+y^2}{p}} - \frac{F}{2}$; therefore the whole Fluent $z = \frac{1}{2}y\sqrt{\frac{pp+yy}{p}} + \frac{2.30258 \text{ log } y + \sqrt{p^2+y^2}}{2}$.

PROB. XVIII.

Required the ſolidity and curve ſuperficies of the ſolid generated by the arc LM, *of the parabola, revolving round* MN *as an axis.*

58. Put $a = \text{VO}$, $b = 2\text{LO}$, $d = \text{NO} = n\text{G}$, $x = \text{M}n$, $y = mn$; then by the nature of the curve $\frac{bb}{a} : y + 2d :: d : x$; therefore $x = \frac{ady + 2add}{bb}$,

in Fluxions $\dot{x} = \frac{ad\dot{y}}{bb}$, hence $cyy\dot{x}$ (the general expreſſion for the cubation of all ſolids) $= \frac{acdy^2\dot{y}}{bb}$ $= \dot{S}$, whoſe Fluent is $\frac{acd}{3bb} \times y^3 = S$, the ſolidity ſought. Fig.

For the Curve Superficies.

Let every thing ſtand as before, only let C = ſuperficies ſought, then becauſe $\dot{x} = \frac{ad\dot{y}}{bb}$, $\dot{x}^2 = \frac{a^2d^2\dot{y}^2}{b^4}$ therefore $\dot{z}$ $(= \sqrt{\dot{x}^2 + \dot{y}^2} =) = \frac{\dot{y}}{b^2}$ $\sqrt{a^2d^2 + b^4}$, and $2cy\dot{z} = \frac{2cy\dot{y}}{bb}\sqrt{a^2d^2 - b^4} = \dot{C}$, whoſe Fluent is $\frac{cy^2}{bb}\sqrt{a^2d^2 + b^4} = C$, the curve ſuperficies ſought.

PROB. XIX.

To find the curve ſurface of the ungula of a cylinder, cut off by the oblique plane BF.

Put $2r = VQ = CD$, $v = FD$, $d = BD$, $x =$ 59.
oQ, $y = om = on$, $z =$ arch $mQ = Qn$, $C =$ required ſuperficies; then by ſimilar triangles $v : d :: x : \frac{dx}{v} = BQ$, whoſe velocity $\frac{d\dot{x}}{v}$ drawn into the generating line $2z$ gives $\frac{2d\dot{x}z}{v} = \dot{C}$, therefore $2d\dot{x}z = v\dot{C}$, and $2d\dot{x}z + 2dx\dot{z} = v\dot{C} + 2dx\dot{z}$, whoſe Fluent is $2dxz = vC + 2drz - 2dry$, therefore common Algebra we have $\overline{ry + xz - rz} \times \frac{2d}{v} = C$.

Fig. If the ſection be at the center, then v, x, r, and y, are all equal, therefore $\overline{rr+rz-tz} \times \frac{2d}{v} = 2dr = C$. But if $v = 2r$, that is, if the ſection be made diagonal wiſe, then $dx = C$, that is, the curve ſurface is juſt half that of the whole cylinder; which may be proved from other principles.

PROB. XX.

Let ABDE *repreſent a cylinder, cut by a plane* DC, *dividing the baſe* AB, *viz. making* AC = CB; *then will* $\frac{\overline{AB}|^2 \times BD}{6}$ *be the ſolidity of the ſlice* CBD, *and the area of the plane* ABDE = *convex ſuperficies.*

60. Draw GK parallel to CD, and IK to BD, then are the triangles KGI, and DCB ſimilar; put $m =$ AC, $n =$ BD, $x =$ CG, then $\sqrt{m^2 - x^2} =$ GI, then by ſimilar triangles $m : n :: \sqrt{m^2 - x^2} : \frac{n}{m}\sqrt{m^2 - x^2}$, therefore the area of the triangle GIK $= \frac{n}{2m} \times \overline{m^2 - x^2}$, and multiplied by $\dot{x}$ gives $\frac{n}{2m} \times m^2\dot{x} - x^2\dot{x}$, the Fluxion of the ſolidity, whoſe Fluent is $\frac{n}{2m} \times m^2x - \frac{x^3}{3}$, and when $x = 2m$, it is $\frac{2nm^2}{3} = \frac{\overline{AB}^2 \times BD}{6}$, the area of the *ungula*.

Again, put $y =$ diſtance of any parallel ſection from the plane CBD, then will $n\dot{y}$ repreſent the velocity wherewith the ſurface is generated, whoſe Fluent ny is the area of the ſurface contained between the two ſections, and when $y = 2m$, it is $2mn$ the convex area of the *ungula*, = AB × BD.

PROB.

Fig. 61.

PROB. XXI.

There is a meadow bounded by an abſciſſa, ſemi-ordinate and curve, whoſe equation is $ax = yy$, *and its area a* minimum; *two men* A *and* B *purchaſe it,* A *being to have the triangle* VBP, *and* B *the remainder. Now there are given* AP = 90 *poles, and* AB = 40, *(perpendicular thereto) 'tis required to find the ſhare of each, and to determine the points* V *and* M, *in* PA *prolonged, and* PM *drawn parallel to* AB, *by geometrical conſtruction.*

Let AB $= b = 40$, AP $= c = 90$, VA $= x$, PM $= y$.

1. $ax = bb$, and $a = \frac{bb}{x}$.

2. $ax + ac = yy$.

3. $\overline{x + c} \times y$ is a *minimum* $= \overline{x + c}\sqrt{ax + ac}$, and putting $\frac{bb}{x}$ for a its equal, we have $\overline{x + c}\sqrt{\frac{x+c}{x}} =$ a *minimum*; in Fluxions $\dot{x}\sqrt{\frac{x + c}{x}} - \frac{\frac{c\dot{x} \times \overline{x+c}}{xx}}{2\sqrt{\frac{x+c}{x}}} = 0$, or $2\dot{x} \times \frac{x + c}{x} - c\dot{x} \times \frac{x + c}{xx} = 0$, which being reduced gives $x = \frac{c}{2} = 45$, and $y = b\sqrt{3} = 69.282$. From whence it may be conſtructed thus,

Set off $\frac{1}{2}$AP from A to V, and 3AB from P to *e*, and AB from P to *d*, biſect *de* in *c*, with the radius *cd* and one foot in *c* deſcribe the arch *d*M, thro' VB and M, draw the curve, and the thing is done. The area of the whole parabola is = 6235.38 poles

poles = $\overset{a}{38}\ \overset{r}{3}\ \overset{p}{35}$. And the triangle VBP = 2700 poles = $\overset{a}{16}\ \overset{r}{3}\ \overset{p}{20}$. Conſequently

	A	R	P
B's ſhare =	22	0	15
A's ſhare =	16	3	20

PROB. XXII.

Given the radius of a circle = *a* (= 50) *to find the area of the greateſt inſcribed parabola.*

62. Let AD = 50 = a, parameter = v, abſciſſa CE = x, ſemi-ordinate AE = y, then by nature of the parabola $vx = yy$, and by property of the circle $\overline{2a - x} \times x = yy$; therefore $vx = \overline{2a - x} \times x$, whence $x = 2a - v$, and $y = \sqrt{2av - vv}$, then will the area of the parabola $= \frac{4yx}{3} = \frac{8a - 4v}{3}\sqrt{2av - vv}, = \frac{1}{3}\sqrt{128a^3v + 96av^3 - 192a^2v^2 - 16v^4}$ = *maximum*, or $128a^3v + 96av^3 - 192a^2v^2 - 16v^4$ = *maximum*, which in Fluxions is $128a^3\dot{v} + 288av^2\dot{v} - 384a^2v\dot{v} - 64v^3\dot{v} = 0$, which divided by $4\dot{v}$ is $32a^3 + 72av^2 - 96a^2v - 16v^3 = 0$, which will be revolved into theſe factors $\overline{a - 2v} \times 8 \times \overline{2a - v} \times \overline{2a - v} = 0$, hence it is evident the parabola will vaniſh when $2a - v = 0$, and be the greateſt when $a - 2v = 0$, therefore $v = \frac{1}{2}a = 25$ the parameter, $x = \frac{3a}{2} = 75$ the abſciſſa, $y = \frac{a}{2}\sqrt{3} =$ 43.30127 the ſemi-ordinate, and the area of the parabola = 4330.127.

PROB.

PROB. XXIII.

Given $ax^m = y^{m+n}$ *expreſſing the nature of a curve, to find a general expreſſion for the value of* AE, *when* EP *is the altitude of the greateſt cylinder, that can lie in a ſpace deſcribed by the rotation of* ADM, *upon its axis* AP.

Let AP $= d$, PM $= b$, AE $= x$, DE $= y$, then EP $= d - x$; then by the nature of the curve $x = \frac{y^{\frac{m+n}{m}}}{a^{\frac{1}{m}}}$, hence the ſolidity of the cylinder is $= 3.1416y^2 \times \overline{d-x} = 3.1416 \times d - \frac{y^{\frac{m+n}{m}}}{a^{\frac{1}{m}}} =$ *maximum*, or $da^{\frac{1}{m}}y^2 - y^{\frac{3m+n}{m}} =$ *maximum*, in Fluxions $2da^{\frac{1}{m}}y\dot{y} - \frac{3m+n}{m}\dot{y}y^{\frac{2m+n}{m}} = 0$, whence $y = \overline{\frac{2m}{3m+n}}\Big|^{\frac{m}{m+n}} \times b$, and $x = \frac{2md}{3m+n} =$ AE, as required. 63.

COROLLARY.

Let AP $= 30$; now if $m = 1$, $n = 2$; then $x = 12 =$ AE and EP $= 18$.

If the figure be a cone, then $m = 1$, $n = 0$; and $x = 20 =$ AE, and EP $= \frac{1}{3}$AP $= 10$. If the *Apollonian* parabola, then $m = 1$, $n = 1$; and $x = 15 =$ AE and EP $= \frac{1}{2}$AP $= 15$.

PROB. XXIV.

Given $yy + x^4 - x = 2yxx$, *an equation expreſſing the nature of a curve, to find its area, when* $y = 260$, *and* $x = 16$.

Firſt $yy + x^4 - x = 2yxx$, by tranſpoſition $yy - 2xxy = x - x^4$, compleating the ſquare is $yy - 2xxy + x^4 = x$, whoſe root is $y - xx = \sqrt{x}$, whence $y = \sqrt{x} + xx$, which multiplied by $\dot{x}$ is $y\dot{x} = x^{\frac{1}{2}}\dot{x} + x^2\dot{x}$ the fluxion of the area whoſe fluent is $\frac{2}{3}x^{\frac{3}{2}} + \frac{x^3}{3} = 1408$.

Note, When x is aſſigned then y cannot be taken at pleaſure.

PROB. XXV.

Suppoſe the earth's radius 3985 *miles, and that* ABC *is a mountain (formed by the ſemi-rotation of the catenaria inverted) whoſe height* BE *is ſuch, that a royal pendulum, which meaſures true time, at* (A) *the foot of the mountain being carried to the top, will in* 24 *hours vary* 1′ 41″ 24‴ *from the truth; and a cannon ball, whoſe greateſt horizontal random is* = BE, *being projected from the top at* B, *with ſuch a direction as to carry it the fartheſt poſſible, will fall at the foot of the mountain* A. *Required the length of both curves, the ſuperficial area of the mountain, the velocity of* B *and* A, *and the angle of direction.*

Firſt, to find the height of the mountain.

Let $r = 3985$ miles the earth's radius, $m = 24 \times 60 \times 60 = 86400''$, $t = 86400'' - 1'\ 41''\ 24''' = 86298''\frac{3}{5}$, then $\frac{rm}{t} - r = 4.67684$ miles the mountain's height = BE.

Or ſince the number of oſcillations in each place being equal, they will be as the time in one place to the time in the other, *viz.* 1440 to 1441.69. Let $r = 3985$ as above, $x =$ radius of the earth and the mountain's height; the force of gravity being as the ſquare of the diſtance from the center, we ſhall have $1440x = 1441.69r$, therefore we ſhall have $x = \frac{1441.69r}{1440} = 3989.67684$ and $3989.67684 - 3985 = 4.67684$ miles the mountain's height $=$ BE, as before.

Let B be the focus of the parabola EDGA, and take $BD = \frac{1}{2}BE$, then will the curve DGA cut the line EA in the required point A. 64.

Put $BE = 4.67684$ miles $= b$, then $BD = \frac{1}{2}b = \frac{1}{4}$ parameter (by property of the parabola) correſponding to the diameter DBE; $EA = y$, then by property of the curve $yy = 3b^2$, hence $y = b\sqrt{3} = 8.100525$ miles $=$ EA, half the mountain's baſe.

Put $c = 16\frac{1}{12}$ feet, $BD = 12346.8576$ feet $= b$, then $2\sqrt{bc} = 891.24323$ feet, the projectile's velocity at B. Again, ſay as $\sqrt{1} : 2\sqrt{bc} :: \sqrt{3} : 2\sqrt{3bc} = 1543.67856$ feet, the projectile's velocity at A.

By the queſtion, B*m* will biſect the angle ABD, and by Trigonometry the angle GB*m* $= 29° \; 56' \; 46''$, the required angle of direction.

Put $BE = x$, $EA = y$, $z =$ the arc B*n*A, then by nature of the *catenaria* (ſee Prob. 13.) $y = 2.30258a$ log. $\frac{a + x + \sqrt{2ax + xx}}{a}$; whence $a = 7.688766$, therefore $z = \sqrt{2ax + xx} = 9.68458$ miles the length of the *catenarian* arch B*n*A, the double of which gives the whole length of the *catenarian* arch 19.36916 miles.

Fig. Put $c = 3.1416 \times 2$, then by *Emerſon*'s Fluxions, $cyz - cax = 266.9797$ ſquare miles, the area of the mountain.

Laſtly, to find the length of the parabolic curve deſcribed by the projectile.

65. Let EA $= 9.35332$ miles $= a$, $c =$ coſine $\angle$BDC, CD $= x$, DB $= y$, curve CB $= z$, then will $\dot{z} = \sqrt{\dot{x}^2 - 2c\dot{x}\dot{y} + \dot{y}^2}$; but the curve CBA is a parabola, whoſe equation is $2bx = yy$, in Fluxions is $2b\dot{x} = 2y\dot{y}$, whence $\dot{x} = \frac{y\dot{y}}{b}$, and by ſubſtitution above, $\dot{z} = \frac{\dot{y}\sqrt{b^2 - 2bcy + y^2}}{b}$, which by putting $v = y - cb$, and $m = b\sqrt{1 - c^2}$, will be transformed to $\frac{\dot{v}\sqrt{m^2 + v^2}}{b}$, whereof the correct Fluent will be $\frac{v\sqrt{m^2 + v^2}}{2b} + \frac{cb}{2} + \frac{m^2}{2b} \times 2.30258 \log. \frac{v + \sqrt{m^2 + v^2}}{b - cb}$, from whence by taking $y = a$,) the length of the curve CBA deſcribed by the projectile, is 10.5174 miles.

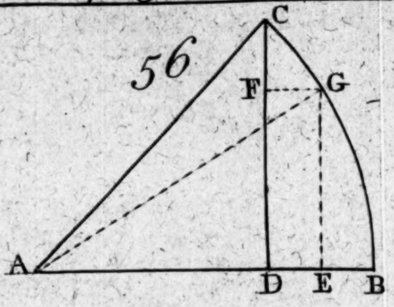
N. B. In the above ſolution, no regard is had to the mountain's attraction, the reſiſtance of the air, and to the ſpherical figure of the earth.

PROB. XXVI.

Given the altitude of a cone $= a$, and the diameter of the baſe $= c$, to find where it muſt be cut by a plane parallel to the baſe, that the area of the ſame may be a mean proportional between the convex ſurface of the fruſtum of a cone, and that of the greateſt cylinder that can be inſcribed in the top part cut off.

66. If $a =$ altitude $=$ VP, and $c =$ diameter $=$ HG, $b =$ PG, $m = 3.1416$, and for PD put x, then

56



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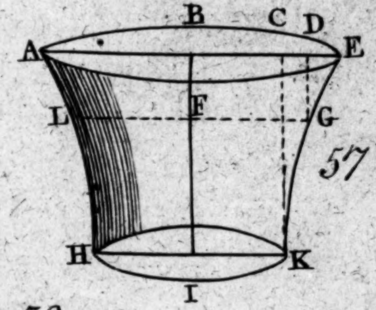

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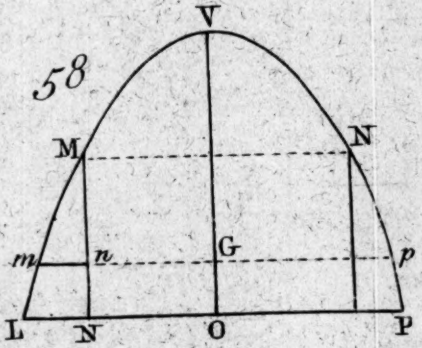

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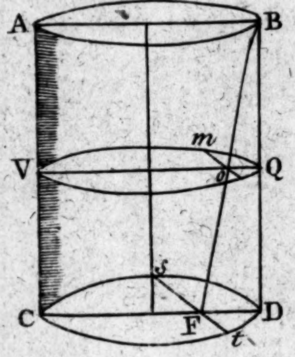

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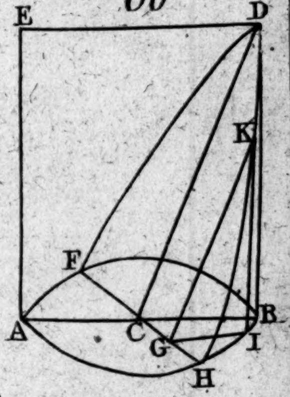

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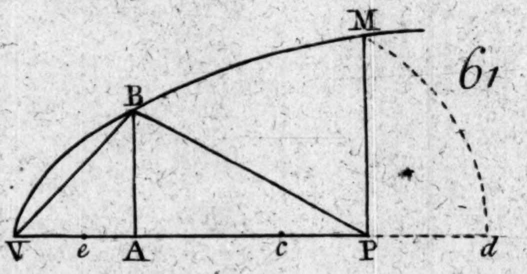

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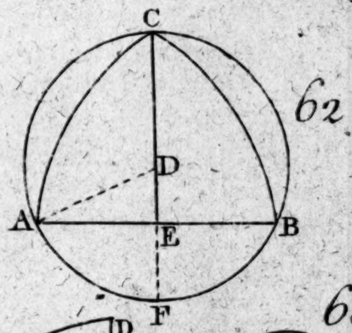

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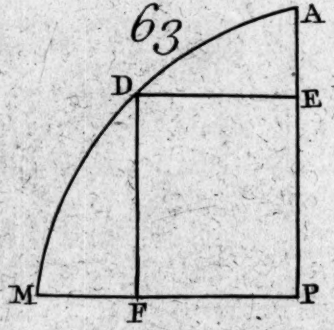

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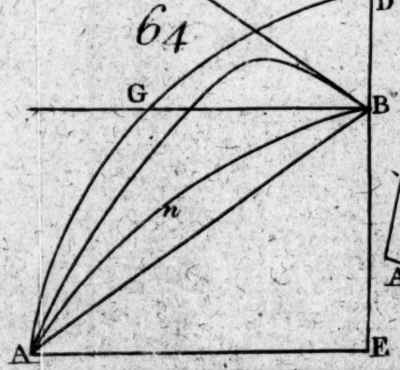

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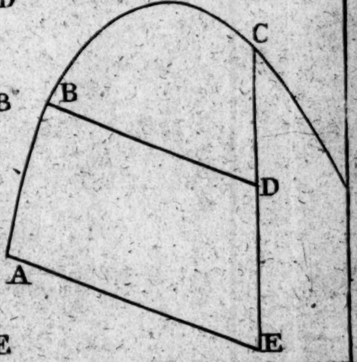

$VD = a - x$; and $a : c :: x : \frac{cx}{a} =$ CE the diameter of the plane, and $\frac{x}{3} = Dn$, the altitude of the cylinder, as was proved *Ex.* 9. *de maximis et minimis.* Then $x : \frac{cx}{a} :: \frac{2x}{3} (Pn) : \frac{2cx}{3a} =$ AB the diameter of the cylinder; therefore $\frac{2cx}{3a} \times \frac{x}{3} \times m = \frac{2cmxx}{9a} =$ ſuperficies of the cylinder, then $\frac{c^2x^2}{aa} \times \frac{m}{4} =$ ſuperficies of the plane.

Again, $a : b :: a - x : \frac{b}{a} \times \overline{a - x} =$ EG, therefore $\frac{cx + ac}{a} \times \frac{m}{2} \times \frac{b}{a} \times \overline{a - x} =$ convex ſurface of the fruſtum, conſequently $\frac{c^4m^2x^4}{16a^4} = \frac{2cmx^2}{9a} \times \frac{cx + ac}{a} \times \frac{m}{2} \times \frac{b}{a} \times \overline{a - x}$ *per* queſtion, hence we have,

Theorem 1. $\frac{64cma^2b}{162c^2 + 288ab} =$ ſuperficies of the cylind.

Theorem 2. $\frac{72c^2mab}{162c^2 + 288ab} =$ ſuperficies of the plane.

Theorem 3. $\frac{81c^3mb}{162c^2 + 288ab} =$ ſuperficies of the cone's fruſtum.

PROB.

PROB. XXVII.

Given $x^{x} = a^{\frac{y}{x}}$, *an equation to an exponential curve; to find the area, when* $x = 10$, *and* $a = 100$.

Put $X =$ hyperbolical logarithm of x, and $A =$ hyp. log. of a, then will $xX = \frac{y}{x} A$ by the nature of exponentials, therefore $y = \frac{x^{2}X}{A}$, which being multiplied by $\dot{x}$ is $y\dot{x} = \frac{\dot{x}x^{2}X}{A}$ the Fluxion of the variable area, whoſe Fluent is, by *Ex.* 43. pa. 64, *Emerſon*'s Fluxions, $= \frac{3x^{3}X - x^{3}}{9A}$, but x is given $= 10$, and $X = 2.302585$ its hyperbolic logarithm, then $\frac{3x^{3}X - x^{3}}{9A} = \frac{3000 \times 2.302585 - 1000}{2.302585 \times 2 \times 9} = \frac{5907.755}{41.44653} = 142.539$ the required area.

PROB. XXVIII.

Given the baſe of a parabola $= 80$, *and its altitude* $= 50$, *required the neareſt diſtance from the middle of the baſe to the curve.*

67. Draw the ſemi-ordinate BC, let BE be the diſtance required.

Put $50 = a$, $40 = b$, $AC = x$, then will $CE = a - x$; by nature of the curve we have, $a : b^{2} :: x : \frac{b^{2}x}{a}$ and $\sqrt{\frac{b^{2}x}{a}} = BC$, and by Theorem 11, of my Introduction to Fortification and Gunnery, $\sqrt{\frac{b^{2}x}{a} + a^{2} - 2ax + xx} = BE$, a *minimum*; in

Fluxions $\frac{b^2\dot{x}}{a} - 2a\dot{x} + 2x\dot{x} = 0$, expunging $\dot{x}$ is $\frac{b^2}{a} - 2a + 2x = 0$, whence $x = a - \frac{bb}{2a} = 34$, and BE $= \frac{b}{2a}\sqrt{4a^2 - b^2} = 36.66$.

PROB. XXIX.

Given AE *the ſide of the parallelogram* BHGEIAB $= 18$ *chains* (*per* Gunter) *and the area equal to the area of the ſquare* CDGEIHC; *it is required to find the greateſt number of acres that can be contained in the parallelogram* CDGHC.

Put AE $=$ BG $= a = 18$, $x =$ GE, then will ED or HG $= \sqrt{ax}$ by the queſtion, and therefore DG will be $= \sqrt{ax - x}$, conſequently $ax - \sqrt{ax^3}$ = area of CDGHC, which muſt be a *minimum*, in Fluxions is $a\dot{x} - \frac{3ax^2\dot{x}}{2\sqrt{ax^3}} = 0$, dividing by $\dot{x}$ is $a - \frac{3ax^2}{2\sqrt{ax^3}}$, whence $2a\sqrt{ax^3} = 3ax^2$, and $x = \frac{4a}{9} = 8$, ſo that $\sqrt{ax} = 12$ chains, $=$ ED &c. and therefore the *maximum* is $\frac{4}{27}aa = 48$ ſq. chains $=$ 4 A 3R 8P. 68.

PROB. XXX.

To find the nature of the curve that will cut all parabolas (*that have the ſame vertex and axis*) *at right angles.*

Let FCE be the curve required, and let the two parabolas AC, A*c*, be infinitely near, C*c* a parti- 69.

cle

Fig. cle of the curve between them, CT a tangent. Draw BC, *bc*, ordinates to the parabolas, and DC *dc* ordinates to the curve.

Let AB or DC $= x$, BC or AD $= y$, ED $= v$, AE $= d$, $oc = \dot{x}$, $o\mathrm{C} = -\dot{y} = \dot{v}$ and $y = d - v$. Draw the tangent CQ, then $\overline{c\mathrm{C}}$ is perpendicular to CQ, or to the parabola at C. Therefore the triangles C*oc*, CQB, are ſimilar; and BQ $= 2x$, by nature of the parabola, ſee *Ex.* 3, *Prob.* 2, for drawing tangents. Therefore QB : BC : : *o*C : *oc*; that is, $2x : d - v :: \dot{v} : \dot{x}$, whence $2x\dot{x} = d\dot{v} - v\dot{v}$, and the Fluent is $xx = dv - \frac{1}{2}vv = \frac{1}{2} \times \overline{2dv - vv}$, or $\overline{\mathrm{DC}}^2 = \frac{1}{2} \times \overline{2dv - vv}$, and equation to an ellipſis.

Therefore if AE $= 10$, AF $= \dfrac{10}{\sqrt{2}} = 7$ nearly.

PROB. XXXI.

10. Given $y = rxzz = 1.5xz^2$ where z is an arc A*m* of the ſemicircle A*m*B, $y =$ any variable ordinate PM of the curve AMN, and $x =$ its abſciſſa AP; to find the area of the curve ſpace APM, (when AB $= 20$, and AP $= 6$,) by a general Theorem.

Firſt, there is given $y = rzzx$, which multiply by $\dot{x}$ is $y\dot{x} = rz^2x\dot{x}$; to find the Fluent, let $a =$ diameter, then $\dot{z} = \dfrac{a\dot{x}}{2\sqrt{ax - xx}} = \dfrac{ax^{-\frac{1}{2}}\dot{x},}{2\sqrt{a - x}}$; by Rule 8, find the Fluent thus, aſſume Fl. $y\dot{x} = \dfrac{rz^2x^2}{2} - s$, whence $\dot{s} = rx^2z\dot{z} = \dfrac{rax^{\frac{3}{2}}z\dot{x}}{2\sqrt{a - x}}$, let $\mathrm{A} =$ Fl. $\dfrac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a - x}}$, and aſſume $s = \dfrac{raz\mathrm{A}}{2} - t$, whence

$\dot{t} = \frac{raA\dot{z}}{2}$, but by 10 and 11 forms, $A = \frac{3az}{4} - \frac{3a+2x}{4}\sqrt{ax-xx}$; whence $\dot{t} = \frac{raA\dot{z}}{2} = \frac{3a^2rz\dot{z}}{8} - \frac{ra\dot{z}}{2} \times \frac{3a+2x}{4}\sqrt{ax-xx} = \frac{3a^2rz\dot{z}}{8} - \frac{ra^2\dot{x}}{4\sqrt{ax-xx}} \times \frac{3a+2x}{4}\sqrt{ax-x\dot{x}} = \frac{3a^2rz\dot{z}}{8} - \frac{3ra^3\dot{x}}{16} - \frac{2a^2rx\dot{x}}{16}$, and $t = \frac{3a^2rz^2}{16} - \frac{3ra^3x}{16} - \frac{ra^2x^2}{16}$, when the area $= \frac{rz^2x^2}{2} - \frac{razA}{2} + t$.

PROB. XXXII.

If the relation between the abſciſſa and ordinate of a curve, be expreſſed by $yx = \overline{a+cx^n}|^m \times dx^{pn}$; *ſuppoſing x the abſciſſa, y the ordinate, to find the area.*

Firſt $y = \overline{a+cx^n}|^m \times dx^{pn-1}$, put $a+cx_n = v$, then $cnx^{n-1}\dot{x} = \dot{v}$. By Rule 8, as in the laſt.

Suppoſe the Fluent of $y\dot{x} = \frac{dv^mx^{pn}}{pn} + q$, then will $\dot{q} = \frac{-mdv^{m-1}x^{pn}\dot{v}}{pn} = \frac{-mdcv^{m-1}x^{pn+n-1}\dot{x}}{p}$; whence $q = \frac{-mcdv^{m-1}x^{pn+n}}{pn.\overline{p+1}} + r$; in Fluxions $\dot{r} = \frac{m.\overline{m-1}.cdv^{m-1}\dot{v} \times x^{pn+n}}{pn.\overline{p+1}} = \frac{m.\overline{m-1}.dc^2nv^{m-2}x^{pn+2n-1}\dot{x}}{pn\,\overline{p+1}}$,

and

Fig. and $r = \frac{m.\overline{m-1}.c^2dv^{m-2}x^{pn+2n}}{pn.\overline{p+1}.\overline{p+2}} + s$, in Fluxions $\dot{s} = \frac{-m.\overline{m-1}.\overline{m-2}.c^2dv^{m-3}x^{pn+2n}\dot{v}}{pn.\overline{p+1}.\overline{p+2}} =$ $\frac{-m.\overline{m-1}.\overline{m-2}.c^3dv^{m-3}x^{pn+3n-1}\dot{x}}{pn.\overline{p+1}.\overline{p+2}}$: Therefore $s = \frac{-m.\overline{m-1}.\overline{m-2}.c^3dv^{m-3}x^{pn+3n}}{pn.\overline{p+1}.\overline{p+2}.\overline{p+3}} + t$, &c. whence the Fluent $y\dot{x} = \frac{dv^m x^{pn}}{pn} - \frac{mcx^n}{\overline{p+1}.v}A -$ $\frac{\overline{m-1}.cx^n}{\overline{p+2}.v}B - \frac{\overline{m-2}.cx^n}{\overline{p+3}.v}C$, &c.

PROB. XXXIII.

Let $x^3+y^3=1000xy$ *expreſs the nature of the curve* ACDE, *with reſpect to the diameter* AB; *it is required to find the area of the greateſt iſoſceles triangle* ACE *that can be inſcribed in the ſaid curve, whoſe baſe* CE *ſhall cut the diameter* AD *at right angles.*

71. Put $1000 = a$, $AE = u$, $FE = z$, and $\sqrt{2} = b$, then if $x^3 + y^3 = axy$ expreſs the nature of the curve with reſpect to the diameter AB, we ſhall have the equation of the curve with reſpect to AD, $bu^3 + 3buz^2 - aaz^2 + az^2 = 0$, conſequently $z = u\sqrt{\frac{a-bu}{a-3bu}}$, and $u^2\sqrt{\frac{a-bu}{a+3bu}}$ will be a *maximum*, which put into Fluxions and reduced, we ſhall have this quadratic equation $u^2 - \frac{a}{3b}u = \frac{aa}{bb}$, where $u = 542.76$, and $z = 143.9$, and the greateſt area will be 78153.

PROB.

Fig.

PROB. XXXIV.

Suppoſe a ball to be projected with a velocity of 400 *feet in a ſecond, at an elevation of* 45 *degrees, and that the reſiſtance of the atmoſphere, at its leaving the mouth of the cannon, is to the force of gravity in a given ratio (ſuppoſe that of equality); to determine the diſtance the ball muſt move, before it arrives at its greateſt altitude, its velocity anſwering to the ſaid altitude, together with the diſtance deſcribed by the ball when its velocity is a minimum; ſuppoſing the law of reſiſtance to be as the ſquare of the celerity.*

As ET : EH : : $\sqrt{2}$: 1 : : 400 : 282.8426 the 72.
velocity in the direction of the horizon, which $=v$, and let $a =$ velocity with which the reſiſtance of the body is equal to its gravity, then by Simpſon's Fluxions, the velocity at the vertex A will be expreſſed by $\frac{av}{\sqrt{a^2+2v^2Q}}$, and the diſtance or arch EA by $\frac{1}{2}d \times 2.30258$ log. $\overline{1+\frac{2v^2Q}{aa}}$, where Q is $= \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2} \times 2.30258 \times$ log. $w + \sqrt{1+w^2}$, (w being the tangent of the given elevation 45 degrees) and $d = \frac{aa}{32\frac{1}{6}}$; therefore in the preſent caſe $2vv$ being $= aa$, and $w = 1$, the velocity at A will become $\frac{v}{\sqrt{1+Q}}$, and the arch EA $= \frac{aa}{64\frac{1}{3}} \times 2.30258 \times$ log. $\overline{1+Q} = 1901.14$.
Let the velocity at A thus found be denoted by c, then the velocity at any other point B, in the di-

rection PB will be expreſſed by $\frac{ac}{\sqrt{a^2+2c^2Q}}$, and the arch AB by $\frac{1}{2}d \times 2.30258 \times \text{log.}\ \overline{1+\frac{2c^2Q}{aa}}$, w being the tangent of the angle ABP, and Q as above; whence $1 : \sqrt{1+w^2} :: \frac{ac}{\sqrt{a^2+2c^2Q}} : \frac{ac\sqrt{1+w^2}}{\sqrt{a^2+2c^2Q}}$ the abſolute velocity at B, which will be a *minimum*, when $\frac{aa+2ccQ}{1+w^2}$ is a *maximum*. Therefore its Fluxion $2c^2\dot{Q} \times \overline{1+w^2} - 2w\dot{w} \times \overline{a^2+2c^2Q} = 0$; or $\frac{\overline{1+w^2}|^{\frac{3}{2}}}{w} = \frac{aa}{cc} + 2Q = \frac{aa}{cc} + w\sqrt{1+w^2} + 2.30258 \times \text{log.}\ w + \sqrt{1+w^2}$, and conſequently $\frac{\sqrt{1+w^2}}{w} - 2.30258 \times \text{log.}\ w + \sqrt{1+w^2} = \frac{aa}{cc} = 4.29$, whence $w = .227 =$ tangent of 12° 48′; therefore the leaſt velocity is $= 188.24$, the diſtance deſcribed from the vertex $= 239.25$, and the whole diſtance deſcribed $=$ 214030.

PROB. XXXV

To find the ſolidity and convex ſurface of a ſemi-cubical paraboloid, the abſciſſa and ſemi-ordinate of whoſe generating parabola is x and y reſpectively.

The equation of the curve is $ax^2 = y^3$, whence $y = \overline{ax^2}|^{\frac{1}{3}}$, and $y^2 = \overline{ax^2}|^{\frac{2}{3}}$, conſequently $c\dot{x} \times \overline{ax^2}|^{\frac{2}{3}} = c\dot{x}yy = \dot{S}$ the Fluxion of the ſolid; that is, $ca^{\frac{2}{3}}$ ×

$\times \dot{x}x^{\frac{4}{3}} = \dot{S}$, whoſe Fluent is $ca^{\frac{2}{3}} \times \frac{3x^{7}}{7}$, or by putting $\overline{\frac{y^{3}}{xx}}\Big|^{\frac{2}{3}}$ for $a^{\frac{2}{3}}$, we have $c \times \overline{\frac{y^{3}}{xx}}\Big|^{\frac{2}{3}} \times \frac{3}{7}\, x^{\frac{7}{3}} = cyy \times \frac{3}{7}\, x = S$, the ſolidity; that is, the ſemicubical paraboloid is $\frac{3}{7}$ of the circumſcribing cylinder, as has been ſhewn another way, *Ex.* 6.

Alſo from the equation of the curve, $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, which in Fluxions is $\dot{x} = \frac{3\dot{y}y^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$, ſquared is $\dot{x}\dot{x} = \frac{9\dot{y}\dot{y}y}{4a}$, then $\dot{x}\dot{x} + \dot{y}\dot{y} = \frac{9y\dot{y}^{2} + 4a\dot{y}^{2}}{4a}$; therefore $\sqrt{\dot{x}^{2} + \dot{y}^{2}} = \frac{\dot{y}}{2a^{\frac{1}{2}}}\sqrt{9y + 4a}$, conſequently $2cy \sqrt{\dot{x}^{2} + \dot{y}^{2}}$ $(= \dot{C})$ is $\frac{cy\dot{y}}{a^{\frac{1}{2}}}\sqrt{9y+4a}$, the Fluxion of the curve ſurface required, whoſe Fluent is $\frac{c}{a^{\frac{1}{2}}} \times$: $\frac{2B^{5}}{5\cdot9\cdot9} - \frac{8aB^{3}}{3\cdot9\cdot9} + Q$ (ſome conſtant quantity) $=$ C, where $B = \sqrt{9y+4a}$, but when $y = 0$, $C = 0$, and $B = 2a^{\frac{1}{2}}$, therefore the correct Fluent is $\frac{c}{\sqrt{a}} \times \overline{\frac{2B^{5}}{5\cdot9\cdot9} - \frac{8aB}{3\cdot9\cdot9}} + \frac{2.8.8caa}{3\cdot5\cdot9\cdot9} = C$ the curve ſurface of the ſemi-cubical paraboloid.

Now if $x = 8$, and $y = 12$, then the ſolidity of the paraboloid will be $= 1551$ nearly, and the convex ſuperficies $= 582.489$, as has been ſhewn another way before.

PROB. XXXVI.

In a parabolic conoid, whoſe altitude is 9, and baſe 6 inches, be cut by a right line at ſome diſtance from, but parallel to its axis; what is the ſolidity and convex ſurface of that ſegment, or part cut off, when the height of the plane of that ſection is 5 inches.

1. For the Solidity.

73. Let AC $= b$, CD $= b$, $a =$ *latus rectum*, pn or $nq = y$, D$n = v$. Then $yy = \mathrm{B}n \times \mathrm{D}n = a \times \mathrm{M}n$, becauſe $p\mathrm{M}q$ is a parabola, whoſe *latus rectum* is a, and $\mathrm{M}n = \frac{yy}{a}$, and the area $p\mathrm{M}q = \frac{4}{3}y \times \mathrm{M}n = \frac{4y^3}{3a}$, but $\dot{v} = \frac{-y\dot{y}}{\sqrt{bb-yy}}$, whence $\frac{4y^3\dot{v}}{3a} = \frac{4y^4\dot{y}}{3a\sqrt{bb-yy}}$ = Fluxion of the ſolidity. But by Form 10, Fl. $\frac{\dot{y}}{\sqrt{bb-yy}} = .017453 \times$ degrees in the arch whoſe ſine is $\frac{y}{b} = \frac{\text{arch D}q}{b} = \phi$. Whence by Form 11, Fl. $\frac{4}{3a} \times \frac{y^4\dot{y}}{\sqrt{bb-yy}} = \frac{4}{3a} \times : \frac{3}{8}b^4\phi - \frac{2y^3+3bby}{8}\sqrt{bb-yy}$, that is, (by reducing, and expunging bb) $= \frac{b \times \text{area } p\mathrm{D}q}{2} - \frac{y^3}{3a}\sqrt{bb-yy}$.

2. For the Curve Surface.

74. Let AB $= b$, CG or HI $= c$, CD $= z$, AI $= x$, ID $= y$, $ax = yy$, $\frac{3.1416}{2} = n$, arch DE $=$ A, then the Fluxion of the ſurface CDE is $= 2\mathrm{A}\dot{z}$, but by the nature of the parabola $\dot{z} = \frac{\dot{y}}{a}\sqrt{aa+4yy}$, ſee *Ex.* 2. *Prob.* 3. And the arch whoſe radius is y and coſine c, is $\mathrm{A} = ny - c - \frac{c^3}{6y^2} - \frac{3c^5}{40y^4}$, &c. by *Prob.*

7, there-

Fig.

7, therefore the Fluxion of the ſurface is

$$-\frac{2\dot{y}}{a}\sqrt{a^2+4y^2}\times ny - c - \frac{c^3}{6yy} - \frac{3c^5}{40y^4}\ \&c. = \frac{2ny\dot{y}}{a}$$

$$\sqrt{a^2+4yy} - \frac{2\dot{y}}{a}\times : 2y + \frac{aa}{4y} - \frac{a^4}{64y^3} + \frac{3a^6}{1536y^5}$$

$$\&c. : \times : c + \frac{c^3}{6y^2} + \frac{3c^5}{40y^4} + \&c. = \frac{2nyy\dot{}}{a}$$

$$\sqrt{aa+4y^2} - \frac{2\dot{y}}{a}\times : 2cy + \frac{caa}{4y} - \frac{ca^4}{64y^3}\ \&c.$$

$$+ \frac{2c^3}{6y} + \frac{a^2c^3}{24y^3}\ \&c.$$

$$+ \frac{6c^5}{40y^3}\ \&c.$$

whoſe Flu. is $= \frac{n}{6a}\times\overline{aa+4yy}|^{\frac{3}{2}} - \frac{2cy^2}{a} - 2.30258$

$\times\overline{\frac{2ca}{4} + \frac{4c^3}{6a}}\times \log. y - \frac{ca^3}{64y^2} + \frac{ac^3}{24y^2} + \frac{3c^5}{2a^2y^2} + \&c.$

from which take the value of the quantity when $y=c$, the remainder is the ſurface CDE.

PROB. XXXVII.

Suppoſe ABGDA *to be a ſpheroid; it is required to find* S, *the quantity of the ſolid generated by the circular ſegment mnropcm, moving from* BD *towards* A.

Or C *the ſolid generated by the ſegment ndorn moving from* BD *towards* A.

1. To find S.

Put $c=$ CD, $t=$ AC, $s=$ CN $(=cr)$ $z=bc$ 75.
$=cd$, and $y=ro=rn$; then by nature of the circle we have $z^2-s^2=yy$, therefore $z^2=s^2+y^2$, and by the nature of the ellipſis we have $\frac{t}{c}\sqrt{c^2-z^2}$

=

$= Cc$, or by ſubſtitution $\frac{t}{c}\sqrt{c^2-s^2-y^2} = Cc$, put $c^2 - s^2 = d^2$, then $\frac{t}{c}\sqrt{d^2-y^2} = cC$, whoſe Fluxion $\frac{-ty\dot{y}}{c\sqrt{d^2-y^2}}$ drawn into $\frac{2s^3}{y} \times : \frac{y^2}{s^2} + \frac{1}{3} - \frac{s^2}{3.5y^2} + \frac{s^4}{5.7y^4} - \frac{s^6}{7.9y^6}$ &c. $= u$ the area of the generating plane *mnropcm*, gives $\frac{-2sty^2\dot{y}}{c\sqrt{dd-yy}} - \frac{2s^3t\dot{y}}{3c\sqrt{dd-yy}} + \frac{2s^5t\dot{y}}{3.5cy^2\sqrt{dd-yy}} - \frac{2s^7t\dot{y}}{5.7cy^4\sqrt{dd-yy}}$ &c. whoſe Fluent by *Ex*. 9, is $\frac{-2st}{c} \times \frac{dd}{2}\varphi + \frac{styR}{2c} - \frac{s^3t}{3c}\alpha + \frac{s^5t}{3.5c}\beta - \frac{s^7t}{5.7c}\gamma$ &c. When $S = 0$, $d = y$, and $R = 0$, and α or $\varphi =$ quadrantal arch $\frac{90° \times .017453\&c.d}{d} = \frac{3.14159d}{2} = 1.57079d$ &c. for which put Q, and then the correct Fluent is $S = \frac{stdd}{2c}Q - \frac{stdd}{2c}\varphi + \frac{styR}{2c} + \frac{2s^3t}{3c}Q - \frac{2s^3t}{3c}\alpha + \frac{2s^5t}{3.5c}\beta - \frac{2s^7t}{5.7c}\gamma + \frac{2s^9t}{7.9c}\delta$ &c. where $\varphi = \frac{\text{arch}}{d}$ of a circle, whoſe radius is d and ſine y; $R = \sqrt{dd-yy}$, $\alpha = \varphi$.

If the Fluent is required when $y = 0$, or when *r*, *o*, *n*, coincide in M, it may be found by Form 17 of Mr. *Emerſon*'s Table in finite terms. The fluxionary quantity muſt be brought to this

— 1

$\overline{-1+ddy^{-2}}|^{-\frac{1}{2}}\, y\dot{y} \times : \frac{-st}{c} - \frac{s^3ty^{-2}}{3c} + \frac{s^5ty^{-4}}{3\; 5c}$ &c.

and the correſponding quantities being compared with thoſe in the form, there will come out $\varphi \times : \frac{-st}{c} - \frac{2s^3t}{3cdd}$: for the Fluent, where φ or the Fluent of $\overline{-1+ddy^{-2}}|^{-\frac{1}{2}}\, y\dot{y}$ or $\frac{y^2\dot{y}}{\sqrt{dd-yy}}$ was found before to be $\frac{dd}{2} \times$ arch (rad. $= d$, and ſine $= y$) $- \frac{y}{2}$ R, and this laſt Fluent corrected gives $-\frac{dd}{2}Q + \frac{dd}{2}\varphi - \frac{Ry}{2}$, and when $y = 0$, it becomes $-\frac{ddQ}{2}$, therefore the Fluent is $-\frac{ddQ}{2} \times \frac{-st}{c} - \frac{2s^3t}{3cdd} = \frac{ddstQ}{2c} + \frac{s^3tQ}{3c} = \frac{3cc-ss}{6} \times \frac{stQ}{c}$

2. To find C.

Let the quantities be repreſented as before, then will $rd = z - s$, and $br = z + s$, then by nature of the figure we get for the area of the plane $\overline{\overline{4z-4s} \times \sqrt{z^2-s^2}} \times : \frac{1}{3} + \frac{z-s}{1.3.5 \times \overline{z+s}} - \frac{\overline{z-s}|^2}{3.5.7 \times \overline{z+s}|^2} + \frac{\overline{z-s}|^3}{5.7.9 \times \overline{z+s}|^3}$, &c. $= u$, which drawn into the Fluxion of $\frac{t}{c}\sqrt{c^2-z^2} = cC$, viz. $\frac{-tz\dot{z}}{c\sqrt{c^2-z^2}}$ gives $\overline{4z-4s} \times \frac{-tz\dot{z}}{c}\sqrt{\frac{zz-ss}{cc-zz}} \times : \frac{1}{3} +$

$z-s$

Fig. $$\frac{z-s}{1.3.5.\overline{z+s}} - \frac{\overline{z-s}\,|^{2}}{3.5.7.\overline{z+s}\,|^{2}} \text{ \&c.} = \overline{z-s} \times \frac{4t z\dot{z}}{c}$$

$$\sqrt{\frac{zz-ss}{cc-zz}} \times : \tfrac{1}{3} + \frac{1}{1.3.5} \cdot \frac{z-s}{z+s} - \frac{1}{3.5.7} \cdot \overline{\frac{z-s}{z+s}}\,\Big|^{2}$$

&c. whoſe Fluent (by *Ex.* 11.) is $= -32s^{4}tv^{\frac{5}{2}} \times :$

$$\frac{2}{3.5} + \frac{52+10q}{3.5.7}\,v + \frac{1017+364q+105q^{2}}{3.5.7.9}\,v^{2} \text{ \&c.}$$

= C the ſolidity of N*nrod*DN.

PROB. XXXVIII.

Let ABGDA *be a parabolic ſpindle; 'tis required to find* S *the ſolid generated by the ſegment mnropcm, moving from* BD *towards* A.

Or C *the ſolid generated by the ſegment ndorn moving from* BD *towards* A.

1. To find S.

75. Put e=CA, b=BC=CD, $a = \frac{ee}{b}$ = latus rectum of the parabola AM*d*D, s=CN, x=*c*C, and $y = cd = cb$, then by the property of the curve $a : e + x :: e - x : \frac{e^{2}-x^{2}}{a} = y = bc$; and (by Mr. *Brown*'s Theorem) $\frac{2se^{2}-2sx^{2}}{a} \times : 1 - \frac{s^{2}a^{2}}{2.3\times\overline{e^{2}-x^{2}}\,|^{2}}$ $- \frac{s^{4}a^{4}}{2.4.5\times\overline{e^{2}-x^{2}}\,|^{4}}$ &c. $= u$, the area of the generating plane *mnropcm*, which being drawn into the Fluxion of *c*C $= \dot{x}$ is $= \overline{e^{2}-x^{2}} \times \frac{2s\dot{x}}{a} - \frac{s^{3}a\dot{x}}{3.\overline{e^{2}-x^{2}}}$ $- \frac{s^{5}a^{2}\dot{x}}{4.5.\overline{e^{2}-x^{2}}\,|^{3}}$ &c. whoſe Fluent (by *Ex.* 10,) is

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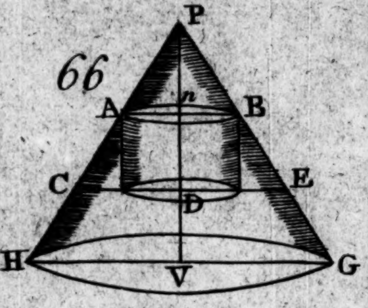
66



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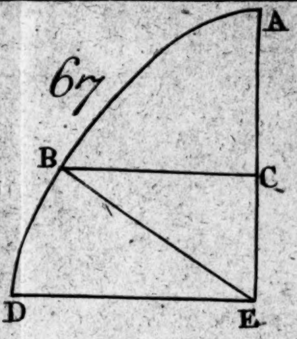

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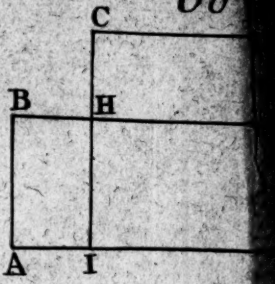

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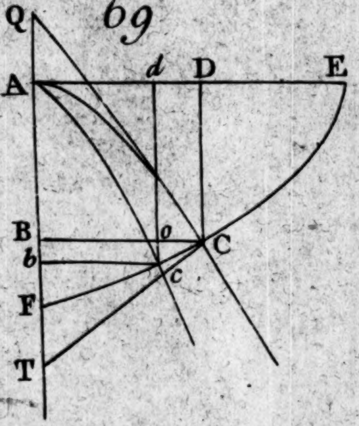

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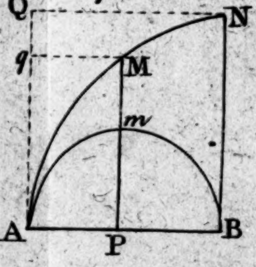

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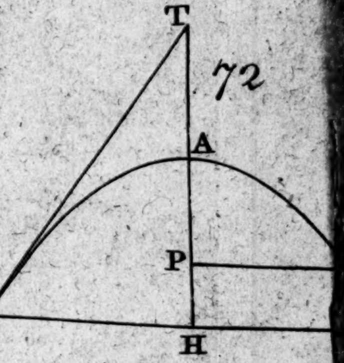

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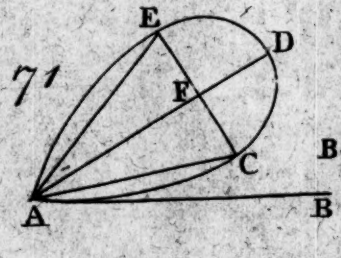

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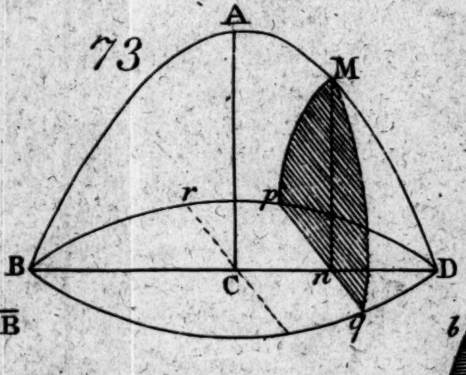

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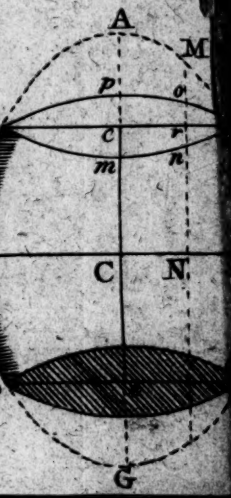

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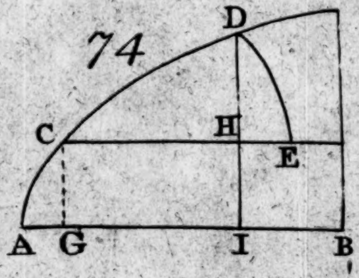

$S = \frac{2se^2x}{a} - \frac{2sx^3}{3a} - \frac{s^3a}{3}\alpha - \frac{s^5a^3}{4\cdot5}\beta$ &c. for the ſolidity of *Cmcporn*NC. Fig.

2. To find C.

Put $s = BN$, $d = DN$, and the other quantities as in the laſt proceſs, then $CN = s - b$, therefore $br =$ (by ſubſtitution) $\frac{be^2 - bx^2}{ee} + s - b = \frac{se^2 - bx^2}{e^2}$; and by nature of the figure $MN = e\sqrt{\frac{d}{b}}$ $(= \sqrt{ad})$ and $a : e\sqrt{\frac{d}{b}} + x :: e\sqrt{\frac{d}{b}} - x : \frac{de^2 - bx^2}{e^2} = rd$.

Then $\frac{\overline{4de^2 - 4bx^2}}{e^4} \times \sqrt{\overline{\overline{se^2 - bx^2} \times \overline{de^2 - bx^2}}} \times : \frac{1}{3} + \frac{de^2 - bx^2}{1\cdot3\cdot5 \times \overline{se^2 - bx^2}} - \frac{\overline{de^2 - bx^2}|^2}{3\cdot5\cdot7 \times \overline{se^2 - bx^2}|^2} + \frac{\overline{de^2 - bx^2}|^3}{5\cdot7\cdot9 \times \overline{se^2 - bx^2}|^3}$ &c. $=$ area of the plane *ndorn*, which being multiplied by $\dot{x}$ is $\overline{de^2 - bx^2}|^{\frac{3}{2}} \times \overline{se^2 - bx^2}|^{\frac{1}{2}} \times \frac{4\dot{x}}{3e^4} + \frac{4}{3\cdot5} \times \frac{\overline{de^2 - bx^2}|^{\frac{5}{2}}}{\overline{se^2 - bx^2}|^{\frac{1}{2}}} \times \frac{\dot{x}}{e^4}$ &c. $= \frac{4\dot{x}}{e^4} \cdot \overline{ae^2 - bx^2}|^{\frac{3}{2}} \cdot \overline{se - bx^2}|^{\frac{1}{2}} \times : \frac{1}{3} + \frac{1}{1\cdot3\cdot} \times \frac{de^2 - bx^2}{se^2 - bx^2} - \frac{1}{3\cdot5\cdot7} \cdot \frac{\overline{de^2 - bx^2}|^2}{se^2 - bx^2}$ &c. whoſe Fluent (by *Ex.* 12.) is $C = \frac{-2e.\overline{s-d}|^3 v^{\frac{5}{2}}}{3\sqrt{d}} \times \frac{2}{5\sqrt{b}} + \frac{2}{5\cdot7} + \frac{s+7d}{7d\sqrt{b}}v$ &c. the ſolidity of *Nnrod*DN.

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